Ch. 6 : The Riemann-Stieltjes Integral

<u>Partition</u>: P of $[a,b]$ is the finite set of points $x_0, x_1 \ldots x_n$ where $a = x_0 \le x_1 \le \cdots \le x_{n-1} \le x_n = b$

$\Delta x_i = x_i - x_{i-1}$ $(i = 1, \ldots, n)$

$M_i = \sup f(x)$ $\quad (x_{i-1} \le x \le x_i)$

$m_i = \inf f(x)$ $\quad (x_{i-1} \le x \le x_i)$

$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$ $\qquad \overline{\int_a^b} f\,dx = \inf U(P,f)$ (upper)

$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$ $\qquad \underline{\int_a^b} f\,dx = \sup L(P,f)$ (lower)

The riemann integral of f over $[a,b]$ is <u>bounded</u>

because it is bounded, there exist m, M s.t.

$m \le f(x) \le M$ $\quad (a \le x \le b)$

So for every $P \Rightarrow m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$

$\Rightarrow$ upper/lower integrals are defined for every bounded function f.

<u>6.2 Def</u> for α, a monotonically increasing function

$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta \alpha_i \Rightarrow \overline{\int_a^b} f\,d\alpha = \inf U(P,f,\alpha)$

$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta \alpha_i \Rightarrow \underline{\int_a^b} f\,d\alpha = \sup L(P,f,\alpha)$

if $\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$ then we can say f is integrable w/ respect to α, we write $f \in \mathcal{R}$

6.3 Def Partition P^* is a refinement of P if $P^* \supset P$. Given two partitions, P_1 & P_2 we say P^* is their <u>common refinement</u> if $P^* = P_1 \cup P_2$

6.4 Thm If P^* is a refinement of P, then

$$L(P, f, \alpha) \le L(P^*, f, \alpha)$$

and

$$U(P^*, f, \alpha) \le U(P, f, \alpha)$$

6.5 Thm

$$\underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha$$

6.6 Thm $f \in \mathcal{R}(\alpha)$ on $[a, b]$ <u>iff</u> for every $\varepsilon > 0$ there exists a partition P such that.

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

6.7 Thm (a) if eq. holds for some P, and some ε, then that equation holds (with the same ε) for every refinement of P.

(b) if the eq above holds for $P = \{x_0, \dots, x_n\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$ then

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta \alpha_i < \varepsilon$$

(c) If $f \in \mathcal{R}(\alpha)$ and the hypothesis of (b) hold, then

$$\left| \sum_{i=1}^{n} f(t_i) \Delta \alpha_i - \int_a^b f\,d\alpha \right| < \varepsilon$$

6.8 Thm | If f is continuous on $[a,b]$ then $f \in \mathscr{R}(\alpha)$ on $[a,b]$

6.9 Thm | If f is monotonic on $[a,b]$, and if α is continuous on $[a,b]$, then $f \in \mathscr{R}(\alpha)$ (assuming α is monotonic)

6.10 Theorem | Suppose f is bounded on $[a,b]$ f has only finitely many points of discontin. on $[a,b]$, and α is continuous at every point at which f is discontinuous.

Then $f \in \mathscr{R}(\alpha)$

6.11 Thm | Suppose $f \in \mathscr{R}(\alpha)$ on $[a,b]$, $m \leq f \leq M$, ϕ is continuous on $[m, M]$ and $h(x) = \phi(f(x))$ on $[a,b]$.

Then $h \in \mathscr{R}(\alpha)$ on $[a,b]$

PROPERTIES OF THE INTEGRAL

6.12 Theorem

(*a*) *If $f_1 \in \mathscr{R}(\alpha)$ and $f_2 \in \mathscr{R}(\alpha)$ on $[a, b]$, then*

$$f_1 + f_2 \in \mathscr{R}(\alpha),$$

$cf \in \mathscr{R}(\alpha)$ for every constant c, and

$$\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha,$$

$$\int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha.$$

(*b*) *If $f_1(x) \le f_2(x)$ on $[a, b]$, then*

$$\int_a^b f_1\, d\alpha \le \int_a^b f_2\, d\alpha.$$

(*c*) *If $f \in \mathscr{R}(\alpha)$ on $[a, b]$ and if $a < c < b$, then $f \in \mathscr{R}(\alpha)$ on $[a, c]$ and on $[c, b]$, and*

$$\int_a^c f\, d\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha.$$

(*d*) *If $f \in \mathscr{R}(\alpha)$ on $[a, b]$ and if $|f(x)| \le M$ on $[a, b]$, then*

$$\left|\int_a^b f\, d\alpha\right| \le M[\alpha(b) - \alpha(a)].$$

(*e*) *If $f \in \mathscr{R}(\alpha_1)$ and $f \in \mathscr{R}(\alpha_2)$, then $f \in \mathscr{R}(\alpha_1 + \alpha_2)$ and*

$$\int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2\,;$$

if $f \in \mathscr{R}(\alpha)$ and c is a positive constant, then $f \in \mathscr{R}(c\alpha)$ and

$$\int_a^b f\, d(c\alpha) = c\int_a^b f\, d\alpha.$$

6.13 Thm) If $f \in \mathscr{R}(\alpha)$ and $g \in \mathscr{R}(\alpha)$ on $[a, b]$, Then

(a) $fg \in \mathscr{R}(\alpha)$;

(b) $|f| \in \mathscr{R}(\alpha)$ and $\left|\int_a^b f\, d\alpha\right| \le \int_a^b |f|\, d\alpha$

6.14 Def| The unit step function I is defined by

$$I(x) = \begin{cases} 0 & (x \le 0) \\ 1 & (x > 0) \end{cases}$$

6.15 Thm) If $a < s < b$, f is bounded on $[a, b]$, f is continuous at s, and $\alpha(x) = I(x-s)$, then

$$\int_a^b f \, d\alpha = f(s)$$

6.16 Theorem) Suppose $c_n \geq 0$ for $1, 2, 3, \ldots$, $\sum c_n$ converges, $\{s_n\}$ is a sequence of distinct points in (a, b) and

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$$

let f be continuous on $[a, b]$. Then

$$\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$$

6.17 Thm) Assume α increases monotonically and $\alpha' \in \mathcal{R}$ on $[a, b]$. Let f be a bounded real function on $[a, b]$

Then $f \in \mathcal{R}(\alpha)$ iff $f\alpha' \in \mathcal{R}$. In that case

$$\int_a^b f \, d\alpha = \int_a^b f(x) \, \alpha'(x) \, dx$$

6.19 Thm) suppose φ is a strictly increasing continuous function That maps an interval $[A, B]$ onto $[a, b]$ Suppose α is monotonically increasing on $[a, b]$ and $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by

$$\beta(y) = \alpha(\varphi(y))$$
$$g(y) = f(\varphi(y))$$

then $g \in \mathcal{R}(\beta)$ and

$$\int_A^B g \, d\beta = \int_a^b f \, d\alpha.$$

Integration & Differentiation

6.20 Thm) Let $f \in \mathcal{R}$ on $[a,b]$ for $a \le x \le b$, put

$$F(x) = \int_a^x f(t)\,dt$$

Then F is continuous on $[a,b]$; furthermore, if f is continuous at a point x_0 of $[a,b]$, then F is differentiable at x_0 and

$$F'(x_0) = f(x_0)$$

6.21 The Fundamental Theorem of Calculus

If $f \in \mathcal{R}$ on $[a,b]$ and if there is a differentiable function F on $[a,b]$ s.t. $F' = f$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

6.22 Integration by parts

Suppose F and G are differentiable functions on $[a,b]$, $F' = f \in \mathcal{R}$ and $G' = g \in \mathcal{R}$ then

$$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx$$