Section 10 - Monotone Seq & Cauchy Seq

Increasing Seq - s_n is increasing seq if $s_n \le s_{n+1}$ $\forall n$

Decreasing Seq - s_n if $s_n \ge s_{n+1}$ $\forall n$

Theorem All bounded monotonic seq converge

Theorem
- If (s_n) is unbounded inc seq $\Rightarrow$ $\lim s_n = +\infty$
- unbounded dec seq $\Rightarrow$ $\lim s_n = -\infty$

$\lim \sup s_n = \lim_{N \to \infty} \sup \{s_n : n > N\}$

$\lim \inf s_n = \lim_{N \to \infty} \inf \{s_n : n > N\}$

s_n has a limit <u>iff</u> $\lim \inf s_n = \lim \sup s_n$

Cauchy Sequence

For each $\varepsilon > 0$, $\exists N$ s.t. $m, n > N \Rightarrow |s_n - s_m| < \varepsilon$

- Convergent seq are Cauchy sequences
- Cauchy seq's are bounded

Theorem A seq is Cauchy <u>iff</u> it is convergent

Section 11 - Subsequences

Subsequence

let $(s_n)_{n \in \mathbb{N}}$ be a seq. subseq is $(t_k)_{k \in \mathbb{N}}$

$\forall k \rightarrow \exists\ n_k \in \mathbb{N}^+$ s.t.

$n_1 < n_2 < \dots < n_k < \dots$ and $t_k = s_{n_k}$

Theorem let (s_n) be a sequence

- If t is in $\mathbb{R}$, $\Rightarrow \exists$ a subseq of (s_n) converging to t iff the set $\{n \in \mathbb{N} : |s_n - t| < \varepsilon\}$ is infinite for all $\varepsilon > 0$.
- If the sequence (s_n) is unbounded above $\Rightarrow$ it has subseq w/ limit $+\infty$
- If (s_n) is unbounded below $\Rightarrow$ a subsequence has limit $-\infty$

Theorem If (s_n) converges $\Rightarrow$ all subseq converge to same limit

Theorem Every (s_n) has a monotonic subsequence

Bolzano - Weierstrass Theorem

Every bounded seq. has a convergent subsequence

Theorem (s_n). There exists monotonic subseq w/ limit = $\lim\sup s_n$

and a monotonic subseq w/ limit = $\lim\inf s_n$

Theorem (s_n) seq in $\mathbb{R}$ & let S be set of subsequential lims of (s_n)

- S is non-empty
- $\sup S = \lim\sup s_n$ and $\inf S = \lim\inf s_n$
- $\lim s_n$ exist **iff** S has one element

Theorem Let S denote the set of subsequential limits of s_n.
Suppose (t_n) is a seq in $S \cap \mathbb{R}$ & that $t = \lim t_n$.
$\Rightarrow$ Then $t \in S$.

Section 12 - lim sup's & lim inf's

$\limsup s_n = \lim_{N\to\infty} \sup\{s_n : n > N\} = \sup S$

$\liminf s_n = \lim_{N\to\infty} \inf\{s_n : n > N\} = \inf S$

Theorem (s_n) converges to pos. real #, s, and (t_n) is any sequence.

$\Rightarrow \limsup s_n t_n = s \cdot \limsup t_n$

Theorem (s_n) seq of non-zero real #'s

$$\Rightarrow \liminf \left|\frac{s_{n+1}}{s_n}\right| \le \liminf |s_n|^{1/n} \le \limsup |s_n|^{1/n} \le \limsup \left|\frac{s_{n+1}}{s_n}\right|$$

Corollary If $\lim \left|\frac{s_{n+1}}{s_n}\right|$ exists (& $= L$)

$\Rightarrow \lim |s_n|^{1/n}$ exists (& $= L$)

Section 13 - Topological Concepts in Metric Spaces

Definition

Let S be a set, & suppose d is a function defined for all pairs (x,y) of elems from S satisfying:

1) $d(x,x)=0 \quad \forall x \in S$ & $d(x,y)>0$ else
2) $d(x,y)=d(y,x) \quad \forall x,y \in S$
3) $d(x,z) \leq d(x,y)+d(y,z) \quad \forall x,y,z \in S$

Definition

- A sequence (s_n) in a metric space (S,d) converges to s in S if $\lim_{n \to \infty} d(s_n, s)=0$
- seq (s_n) is cauchy seq if for each $\epsilon>0$, $\exists N$ s.t.

  $m,n > N \implies d(s_m, s_n) < \epsilon$

Complete metric space is complete if every Cauchy seq in S converges to some element in S.

☆ A Euclidean k-space $\mathbb{R}^k$ is complete

Theorem Every bounded sequence in $\mathbb{R}^k$ has a convergent subseq.

Interior Let (S,d) be a metric space. Let E be a subset of S. An elem $s_0 \in E$ is interior to E if for some $r>0$ we have

$$\{s \in S : d(s_0, s) < r\} \subseteq E$$

E° = set of pts interior to E

Open Set is open if every pt in E is interior $\iff E^\circ = E$

Closed compliment of open set. E is closed if $E = S \setminus U$, when U is open

- collection of closed sets is closed
- intersection of all closed sets = E^-
- Boundary of E is $E^- \setminus E^\circ$

Closed/open Propositions

- The set E is closed iff $E = E^-$
- The set E is closed iff it contains lims of every convergent seq of pts in E
- An el is in E^- iff it is the limit of some seq of points in E
- A point is in the boundary of E iff it belongs to closure of both E & comple

Open cover $\mathcal{U}$ + if each point of E belongs to at least one set in $\mathcal{U}$

Sub cover of $\mathcal{U}$ is any subfamily of $\mathcal{U}$ that also covers E

Compact - E is compact if every open cover of E has a finite subcover

Heine-Borel Theorem

A subset E of $\mathbb{R}^k$ is compact iff it is closed & bounded

Ch 14 Series

Inf Series

partial sum

let $s_n = a_m + \cdots + a_n = \sum_{k=m}^{n} a_k$ so if (s_n) conv to S then

$\sum_{n=m}^{\infty} a_n = S$ $\lim s_n = S$ $\lim_{n\to\infty} \left(\sum_{k=n}^{n} a_k \right) = S$

- $\sum \frac{1}{n^p} \to$ conv iff $p > 1$

- $\sum_{k=0}^{n} ar^k = a \frac{1-r^{k+1}}{1-r}$
- $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ if $|r| < 1$

Def Series $\sum a_n$ satisfies Cauchy Criterion if seq (s_n) of partial sums is a Cauchy seq
for each $\epsilon > 0$ $\exists$ an N s.t. $m, n > N \Rightarrow |s_n - s_m| < \epsilon$

Thrm A series converges iff it satisfies the Cauchy criterion.

Cor If series $\sum a_n$ converges $\Rightarrow \lim a_n = 0$

Comparison Test

Let $\sum a_n$ be a series where $a_n \geq 0$ for all n.

(i) If $\sum a_n$ converges & $|b_n| \leq a_n$ $\forall n$,
$\Rightarrow \sum b_n$ converges

(ii) If $\sum a_n = +\infty$ & $b_n \geq a_n$ $\forall n \Rightarrow \sum b_n = +\infty$

Cor Absolutely convergent series are convergent

Ratio Test

A series $\sum a_n$ of nonzero terms

(i) conv abs if $\lim\sup |a_{n+1}/a_n| < 1$

(ii) diverges if $\lim\inf |a_{n+1}/a_n| > 1$

(iii) Otherwise $\lim\inf |a_{n+1}/a_n| \leq 1 \leq \lim\sup |a_{n+1}/a_n|$
↳ no info

Root Test

$\alpha = \lim\sup |a_n|^{1/n}$

- converges absolutely if $\alpha < 1$
- diverges if $\alpha > 1$
- Otherwise $\alpha = 1$ & test gives no info

Section 15 Alternating Series & Integral Test

Thrm $\sum \frac{1}{n^p}$ converges iff $p > 1$

↳ if $p>1$ =>

$\sum_{k=1}^{n} \frac{1}{k^p} \leq 1 + \int_1^n 1/x^p \, dx = 1 + \frac{1}{p-1}\left(1 - \frac{1}{n^{p-1}}\right) < 1 + \frac{1}{p-1} = p/(p-1)$

Integral Test

If $\lim_{n\to\infty} \int_1^n f(x)dx = \underline{+\infty}$ → diverge $\underline{< +\infty}$ → converge

use when

- terms a_n are non neg
- There is a dec fun f on $[1, \infty)$ s.t $f(n) = a_n$ $\forall n$
- Integral is easy to calculate

Alternating Series Thrm

if $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ & $\lim a_n = 0$ => alternating series $\sum (-1)^{n+1} a_n$ converges

moreover the part sums $s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$ satisfy $|s - s_n| \leq a_n$