Today : $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

- Natural number.

$$\mathbb{N} = \{0, 1, 2, 3, \cdots \}.$$

- successor construction : 2 is the successor of 1.
  3 ——— of 2.

so, starting from 0, one can reach all the natural numbers.

- Peano Axioms for natural number (read: Tao, I)

mathematical induction property (Axiom 5):

Let n be a natural number, $P(n)$ be a statement depending on n, if the following 2 conditions holds.

(a) $P(0)$ is true

(b). If $P(k)$ is true, then $P(k+1)$ is true.
(i.e. $P(k) \Rightarrow P(k+1)$).

then, $P(n)$ is true for all $n \in \mathbb{N}$.

Rmk:

"operations allowed for $\mathbb{N}$" : $+, \times$ ("$\cdot$" also use dot for product)

- if $n, m \in \mathbb{N}$, then $n+m \in \mathbb{N}$, $n \times m \in \mathbb{N}$
- $-, /$ are not always defined.

if unfamily about "set", read Rudin

(2) Integers : $\mathbb{Z} = \{ \cdots -2, -1, 0, 1, 2, \cdots \}$.

allowed operation: $+, -, \cdot$ (no division here).

(formally, $\mathbb{Z}$ is a "ring") (you can read about ring in Harrison Chen's note).

(3). Rational numbers: $\mathbb{Q} = \{ \frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0 \}$.

we have all 4 operations $+, -, \cdot, /$.

$\mathbb{Q}$ is now a "field".

$\mathbb{Q}$ is an ordered field, there is a "relation" $\leqq$.

A relation S is a subset of $\mathbb{Q} \times \mathbb{Q}$, if $(a,b) \in S$, we say "a and b has relation S", or "$a S b$"

The "$\leq$" relation has 3 properties:

(1) $a \leq b$ and $b \leq a$, then $a = b$

(2). if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity).

(3) for any $a, b \in \mathbb{Q}$, we have $\underbrace{\quad a \leq b \text{ or } b \leq a}_{\text{at least one of the following}}$

Ordered Field $\mathbb{Q}$: means the field structure $(+, -, \cdot, /)$ is compatible with $(\leq)$.

(a). if $a \leq b$, then $a + c \leq b + c$ for all $c \in \mathbb{Q}$.

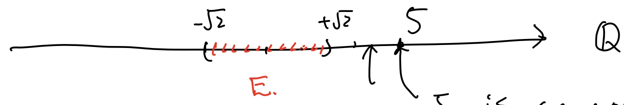
(b) if $a \geq 0$, and $b \geq 0$, then $a \cdot b \geq 0$.

What's lacking about $\mathbb{Q}$?

There are certain gaps in $\mathbb{Q}$: (1). for example, the equation $x^2 = 2$ cannot be solved in $\mathbb{Q}$.

(2) for a bounded subset in $\mathbb{Q}$, call it E, it may not have a "most economical" or "sharpest" upper bound in $\mathbb{Q}$.

e.g.: $E = \{ x \in \mathbb{Q} \mid x^2 < 2 \}$.



5 is an upper bound of E.

4 is also

there is no $\overbrace{\text{least upper bound}}^{\text{sup}}$ of E in $\mathbb{Q}$

(we want to say $\sqrt{2}$ as the sup(E). but $\underline{\sqrt{2} \text{ is not a rational number}}$.)

? $\sqrt{2} \neq \frac{m}{n}$