Math 104. Lec 3 Sequence and Limits.

Ross §9.

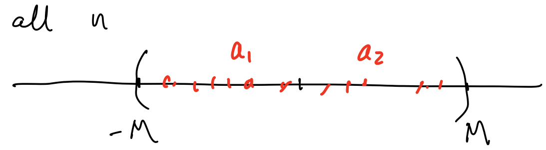
Recall last time:

Def: 1. A seq of real numbers (a_n) converges to $a \in \mathbb{R}$, if for any $\varepsilon > 0$, there exists N (positive integer), such that for all $n > N$, we have $|a_n - a| < \varepsilon$.

Today: properties for convergent seq.

Thm 9.1 Convergent sequences are bounded.

(recall, a seq (a_n) is bounded, if $\exists M > 0$, such that $|a_n| \leq M$ for all n



)

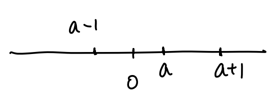
Pf: Let (a_n) be a convergent sequence, that converge to $a \in \mathbb{R}$.

Let $\varepsilon = 1$, then by def of convergence, there exists $N > 0$. s.t.

$\forall n > N$, $|a_n - a| < 1$. $\Leftrightarrow$ $\underline{a-1 < a_n < a+1}$ $\underline{\forall n > N}$

Let $M_1 = \max\{a_1, a_2, \cdots, a_N\}$.

$M_2 = \max\{|a-1|, |a+1|\}$



$M = \max\{M_1, M_2\}$.

Thus, if $n \leq N$, we have $|a_n| \leq M_1$

if $n > N$, we have $|a_n| \leq M_2$

thus. $\forall n$, $|a_n| \leq \max\{M_1, M_2\} = M$.

Moral: one can deal with the first few term of a seq easily, it is the "tail of the seq." that matters.

Operations on convergent seq.:

(1) $\forall\, c \in \mathbb{R}$, $\forall$ convergent seq $a_n \to a$, we have

$$c \cdot a_n \to c \cdot a.$$

Pf: If $c = 0$, the result is obvious.

If $c \neq 0$, we need to show for any $\varepsilon > 0$, $\exists\, N > 0$. s.t. $\forall n > N$, $|c \cdot a_n - c a| < \varepsilon$. But, this is equivalent to $|c| \cdot |a_n - a| < \varepsilon$.

$$\Leftrightarrow \quad |a_n - a| < \frac{\varepsilon}{|c|}.$$

Now, let $\varepsilon' = \varepsilon / |c|$, and using convergence of $a_n \to a$, we have $N > 0$. s.t. $\forall n > N$, $|a_n - a| < \varepsilon' = \frac{\varepsilon}{|c|}$. This gives us the desired N. $\square$.

(2) If $a_n \to a$, $b_n \to b$, then $a_n + b_n \to a + b$.

Pf: We want to show, $\forall\, \varepsilon > 0$, $\exists\, N$. s.t. $\forall n > N$, $|a_n + b_n - (a+b)| < \varepsilon$.

The requirement $\Leftrightarrow$ $|(a_n - a) + (b_n - b)| < \varepsilon$ $\quad (*)$

$\because$ $|(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$ by triangle inequality

$\therefore$ $(*) \Leftarrow |a_n - a| + |b_n - b| < \varepsilon.$ $\quad (**)$

$$\Leftarrow \begin{cases} |a_n - a| < \varepsilon/2 \\ |b_n - b| < \varepsilon/2. \end{cases} \quad (***).$$

By convergences of a_n and b_n, $\exists\, N_1, N_2$, s.t. $\forall n > N_1$, $|a_n - a| < \varepsilon/2$, ; $\forall n > N_2$, $|b_n - b| < \varepsilon/2$.

Take $N = \max\{N_1, N_2\}$, then $\forall n > N$. $(***)$ is satisfied.

hence $(*)$ is satisfied.

Cor : if $a_n \to a$, $b_n \to b$, then $a_n - b_n \to a - b$.

Pf : let $C_n = (-1) \cdot b_n$, then $C_n \to -b$.

then $\overset{a_n - b_n}{\overset{\shortparallel}{a_n + C_n}} \to a - b$. #.

(3) If $a_n \to a$, $b_n \to b$, then $a_n \cdot b_n \to a \cdot b$.

~~Pf~~: WTS : $\forall \varepsilon > 0$, $\exists N$. s.t. $\forall n > N$.

$$|a_n b_n - ab| < \varepsilon. \qquad (*)$$

$$\begin{aligned} |a_n b_n - ab| &= |a_n (b_n - b) + a_n \cdot b - ab| \\ &= |a_n \cdot (b_n - b) + (a_n - a) b|. \\ &\leq |a_n (b_n - b)| + |(a_n - a) b| \\ &\leq |a_n| \cdot |b_n - b| + |a_n - a| \cdot |b|. \end{aligned}$$

$\because$ $a_n \to a$ $\therefore$ a_n is bounded $\therefore$ $\exists M_1 > 0$. s.t. $|a_n| \leq M_1$ $\forall n$.

$$\leq M_1 |b_n - b| + |b| \cdot |a_n - a|.$$

$$(*) \Leftarrow \begin{cases} M_1 |b_n - b| < \frac{\varepsilon}{2} \\ |b| \cdot |a_n - a| < \frac{\varepsilon}{2}. \end{cases} \qquad (**).$$

Since $a_n \to a$, let $\varepsilon_1 = \frac{\varepsilon}{2} / |b|$, then $\exists N_1$, s.t. $\forall n > N_1$,

$$|a_n - a| < \varepsilon_1 \iff |b| \cdot |a_n - a| < \frac{\varepsilon}{2}.$$

Since $b_n \to b$, let $\varepsilon_2 = \frac{\varepsilon}{2} / M_1$. then $\exists N_2$, s.t $\forall n > N_2$,

$$|b_n - b| < \varepsilon_2 \iff M_1 \cdot |b_n - b| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$, thus for all $n > N$,

we have $(**)$ holds, hence $(*)$ holds. $\square$

(4). If $a_n \to a$, and $a_n \neq 0$ $\forall n$, and $a \neq 0$,

then $\frac{1}{a_n} \to \frac{1}{a}$.

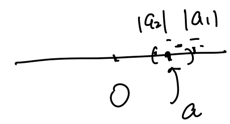
(note, $a_n \neq 0$ does not imply $a \neq 0$,
ex. $a_n = \frac{1}{n}$, $a = 0$.)

Pf: WTS, $\forall \varepsilon > 0$, $\exists N$, s.t. $\forall n > N$.

$$\left| \frac{1}{a_n} - \frac{1}{a} \right| < \varepsilon. \qquad (*)$$

$$\left| \frac{1}{a_n} - \frac{1}{a} \right| = \left| \frac{a - a_n}{a \cdot a_n} \right| = \frac{|a - a_n|}{|a| \cdot |a_n|}$$

Claim: $\exists\, c > 0$, such that $|a_n| > c$ $\forall n$.



Let $\varepsilon' = |a|/2$, then $\exists N'$, s.t.

$\forall n > N'$, $|a_n - a| < \varepsilon' = |a|/2$.

$\Leftrightarrow$ $-|a|/2 < a_n - a < |a|/2$

$\Leftrightarrow$ $\underline{a - |a|/2 < a_n < a + |a|/2}$

$\Rightarrow$ $|a_n| > |a|/2$

If $a > 0$, then $|a| = a$.

$\frac{1}{2}|a| < a_n < \frac{3}{2}|a|$

If $a < 0$, then $a = -|a|$.

$-\frac{3}{2}|a| < a_n < -|a| + \frac{|a|}{2} = -\frac{1}{2}|a|$

$\Rightarrow |a_n| > \frac{1}{2}|a|$.

let $c_1 = \min \{ |a_1|, |a_2|, \cdots, |a_{N'}| \} > 0$.

let $c = \min \{ c_1, |a|/2 \}$. finish proof of claim.

Thus. $\frac{|a_n - a|}{|a| \cdot |a_n|} < \frac{|a_n - a|}{|a| \cdot c}$.

Hence $(*) \Leftarrow \frac{|a_n - a|}{|a| \cdot c} < \varepsilon \quad (**)$.

and $(**)$ can be achieved using $a_n \to a$. $\square$

<u>Cor</u>: if $a_n \to a$, $b_n \to b$, and $b_n \neq 0$, $b \neq 0$.

then $\frac{a_n}{b_n} \to \frac{a}{b}$.

<u>Pf</u>: $\frac{a_n}{b_n} = a_n \cdot \left(\frac{1}{b_n}\right)$. $\because \frac{1}{b_n} \to \frac{1}{b}$ by Lemma.

$\therefore a_n \cdot \left(\frac{1}{b_n}\right) \to a \cdot \frac{1}{b}$. ($\because$ product of convergent seq still converge) $\square$.

<u>Thm 9.7</u>: (Useful Results).

(1) $\lim_{n\to\infty} \frac{1}{n^p} = 0 \quad \forall p > 0$.

(2) $\lim_{n\to\infty} a^n = 0. \quad \forall |a| < 1$

(3) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

<u>sketch</u>: let $S_n = n^{\frac{1}{n}} - 1$, thus $S_n \geq 0 \quad \forall n$ positive integer.

$1 + S_n = n^{\frac{1}{n}} \Leftrightarrow (1+S_n)^n = n$.

using binomial expansion.

$$1 + n \cdot S_n + \frac{n(n-1)}{2} S_n^2 + \cdots\cdots = n$$

$$\Rightarrow \quad \frac{n(n-1)}{2} \cdot S_n^2 \leq n.$$

$$\Rightarrow \quad S_n^2 \leq \frac{2}{n-1}$$

Thus $S_n \to 0$ as $n \to \infty$

(4). $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$ for all $a > 0$.

" $\lim a^{\frac{1}{n}} = a^{\lim \frac{1}{n}} = a^0 = \underline{1}$. "

Ross Ex

Discussion. 9.2, 9.9(c), 9.15.