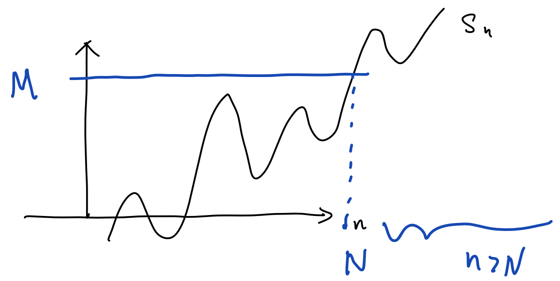
§10 in Ross: monotone seq & lim sup, lim inf.

- **Def** ($\lim s_n = +\infty$). A sequence (s_n) is said to "diverge to $+\infty$", if for any $M \in \mathbb{R}$, there is an N. s.t.

$$s_n > M \qquad \forall\, n > N.$$



- Recall:
  - **Def**: (sup of a set). Given a subset $S \subset \mathbb{R}$. If S is not bounded above, then $\sup S = +\infty$. If S is bounded above, then $\sup S$ is the number r, that is an upper bound, and for any $\varepsilon > 0$, there is some $s \in S$, that $s > r - \varepsilon$ ($r - \varepsilon$ is not an upper bound of S).

- **Def**: ("value set" of a sequence). If $(s_n)_{n=1}^{\infty}$ is a sequence, then $\{s_n\}_{n=1}^{\infty}$, the subset of $\mathbb{R}$ that (s_n) values in, is called the value set.

Ex: • $(s_n) = 1, 2, 1, 2, 1, 2, \cdots$ "journey"
$\{s_n\}_{n=1}^{\infty} = \{1, 2\}$. "foot print"

• $(s_n) = 1, 1, 2, 2, 1, 1, 2, 2, \cdots$
$\{s_n\}_{n=1}^{\infty} = \{1, 2\}$.

- $(S_n) = 1, 2, 3, 4, \cdots$

  $\{S_n\}_{n=1}^{\infty} = \{1, 2, 3, 4, \cdots\}$.

- <u>Def</u> (monotone seq).
  - A seq $(a_n)_{n=1}^{\infty}$ is monotonically increasing, if

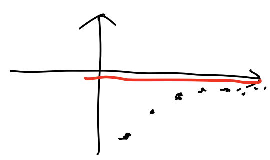
    $a_{n+1} \geq a_n \qquad \forall n = 1, 2, \cdots$

  - A seq (a_n) is monotonically decreasing, if

    $a_{n+1} \leq a_n \qquad \forall n.$

<u>Ex</u>:
- $(a_n) = a$, constant seq is monotone increasing & decreasing
- $(a_n) = 1, 2, 3, \cdots$, is increasing
- $(a_n) = -\frac{1}{n}$

  $n \geq 1$ $\quad \lim a_n = 0$



increasing, and bounded above.

(hence bounded below).

<u>Thm</u>: A bounded monotone sequence is convergent.

<u>Pf</u>: (only prove for increasing seq). Let (a_n) be a bounded monotone seq. Let $\gamma = \sup \underbrace{\{a_n\}_{n=1}^{\infty}}_{\text{a set}}$ $(= \sup_n a_n)$

Then.
- $a_n \leq \gamma \qquad \forall n.$
- for any $\varepsilon > 0$, $\exists\, a_{n_0}$, s.t. $a_{n_0} > \gamma - \varepsilon$.

Thus, for any $\varepsilon > 0$, let $N = n_0$ (n_0 defined above.) then for any $n > N$, we have

$$\gamma - \varepsilon < a_{n_0} \leq a_n \leq \gamma$$

thus $\quad |a_n - \gamma| < \varepsilon.$

Hence $\quad \lim a_n = \gamma.$ $\square$

<u>Ex</u>: (Recursively defined seq.):

Let S_1 be any positive number. Let

$$(*) \qquad S_{n+1} = \frac{S_n^2 + 5}{2 S_n} \qquad \forall n \geq 1.$$

we want show $\lim S_n$ exists and find it.

<u>Rmk</u>: (1) if we assume $\underline{\lim S_n \text{ exist}}$, call it S, then

S satisfies.

$$(**) \qquad S = \frac{S^2+5}{2S}.$$

$\because$ we can apply the operation $\lim\limits_{n\to\infty}(\cdots)$ to both sides of $(*)$

$$(**) \Rightarrow 2S^2 = S^2 + 5 \Rightarrow S^2 = 5 \Rightarrow$$

S can be $\pm\sqrt{5}$. Since S_n is a positive seq.

$\lim S_n$ can only be ≥ 0, thus S can only be $\sqrt{5}$.

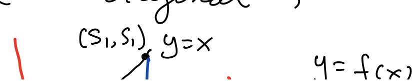
(2) To show $\lim S_n$ exists, we only need to show

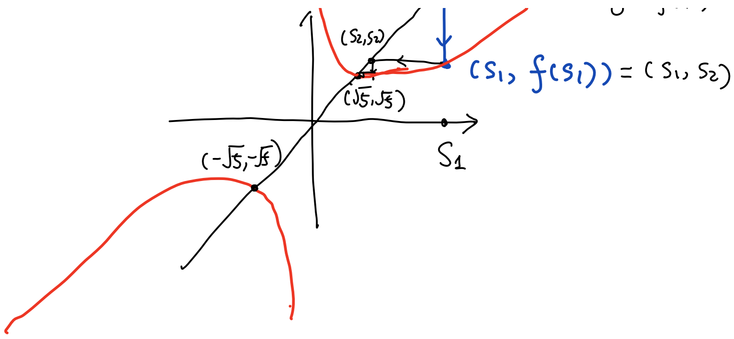
S_n is bounded and monotone.

<u>Here is a trick</u>: • Let $f(x) = \dfrac{x^2+5}{2x}$, then $S_{n+1} = f(S_n)$.

• Consider the graph of f, i.e. $y = f(x)$.

• Consider the diagonal, i.e. $y = x$.

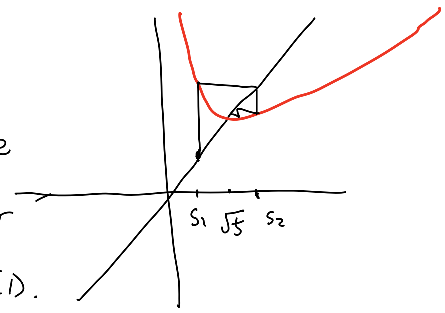




(1)• if $\underline{S_1 > \sqrt{5}}$, we should try to prove.

$\underline{\sqrt{5}} < \cdots S_3 < \underline{S_2} < S_1$

(2)• if $0 < S_1 < \sqrt{5}$, then we have $S_2 > \sqrt{5}$., we can consider $\underline{(S_n)_{n=2}^{\infty}}$, that reduces to case (1).



- If (S_n) is unbounded and increasing, then $\underline{\lim S_n = +\infty}$.
- ——— ——————— decreasing, then $\lim S_n = -\infty$.

§ Lim inf and lim sup. of a sequence.

• Def (lim sup). Let $(S_n)_{n=1}^{\infty}$ be a sequence,

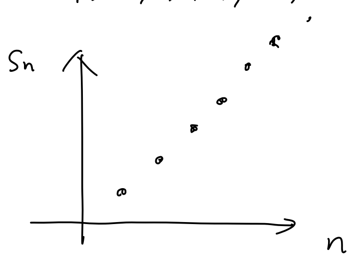
$$\limsup_{n\to\infty} S_n := \lim_{n\to\infty}\left(\sup \{S_m\}_{m=n}^{\infty} \right).$$

notation ~~Def~~: • $(S_n)_{n=N}^{\infty}$ is called a "tail of the seq (S_n)" starting at N.

- $A_N = \sup \{S_n\}_{n=N}^{\infty} = \sup_{n \geq N} S_n.$

- $\limsup S_n = \lim_{N \to \infty} A_N.$

Ex: (1) $(S_n) = 1, 2, 3, 4, 5, \cdots\cdots$

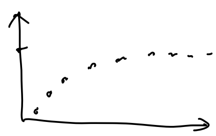


$A_1 = \sup_{n \geq 1} S_n = +\infty$

$A_2 = \sup_{n \geq 2} S_n = +\infty.$

$\limsup S_n = \lim A_N = +\infty$

(2) $(S_n) = 1 - \frac{1}{n}.$



$A_1 = \sup_{n \geq 1} S_n = 1$

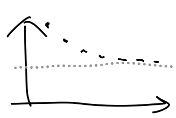
$A_2 = \sup_{n \geq 2} S_n = 1$

$\vdots$ $A_n = 1.$

$\limsup S_n = \lim A_N = 1$

(actually, for any monotone increasing seq, $\limsup S_n = \sup S_n = A_1$)

(3) $S_n = 1 + \frac{1}{n}$



$(S_n) = 2,\ 1+\frac{1}{2},\ 1+\frac{1}{3},\ 1+\frac{1}{4}, \cdots$

$A_1 = \sup \{2,\ 1+\frac{1}{2},\ 1+\frac{1}{3}, \cdots\} = 2$

$A_2 = \sup \{1+\frac{1}{2},\ 1+\frac{1}{3},\ 1+\frac{1}{4}, \cdots\} = 1+\frac{1}{2}$

$\vdots$

$$A_n = S_n = 1 + \frac{1}{n}$$

$$\limsup S_n = \lim \left(1 + \frac{1}{n}\right) = 1.$$

<u>Lemma</u>: $A_n = \sup_{m \geq n} S_m$ forms a <u>decreasing</u> sequence.

<u>pf</u>: since $\{S_m\}_{m=n}^{\infty} \supset \{S_m\}_{m=n+1}^{\infty}$, thus.

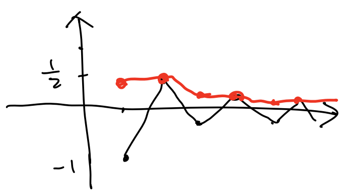
$$\sup \{S_m\}_{m=n}^{\infty} \geq \{S_m\}_{m=n+1}^{\infty},$$

i.e. $A_n \geq A_{n+1}$.

<u>Cor</u>: $\lim_{n\to\infty} A_n = \inf \{A_n\}_{n=1}^{\infty} \quad (= \inf_n A_n)$.

(4). $S_n = (-1)^n \cdot \frac{1}{n}$.

$S_1 = -1, \quad S_2 = \frac{1}{2}, \quad S_3 = -\frac{1}{3}, \quad \cdots$



$$A_1 = \sup_{n \geq 1} (S_n) = S_2 = \frac{1}{2}$$

$A_2 = \frac{1}{2}, \quad A_3 = S_4 = \frac{1}{4}, \quad \cdots$

$(A_n) = \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \cdots$

A_n is like the "upper envelope".

$$\limsup S_n = \lim A_n = 0.$$