## 2

## A Taste of Topology

## 1 Metric Spaces

It may seem paradoxical at first, but a specific math problem can be harder to solve than some abstract generalization of it. For instance if you want to know how many roots the equation

$$
t^{5}-4 t^{4}+t^{3}-t+1=0
$$

can have then you could use calculus and figure it out. It would take a while. But thinking more abstractly, and with less work, you could show that every $n^{\text {th }}$-degree polynomial has at most $n$ roots. In the same way many general results about functions of a real variable are more easily grasped at an abstract level - the level of metric spaces.

Metric space theory can be seen as a special case of general topology, and many books present it that way, explaining compactness primarily in terms of open coverings. In my opinion, however, the sequence/subsequence approach provides the easiest and simplest route to mastering the subject. Accordingly it gets top billing throughout this chapter.

A metric space is a set $M$, the elements of which are referred to as points of $M$, together with a metric $d$ having the three properties that distance has in Euclidean space. The metric $d=d(x, y)$ is a real number defined for all points $x, y \in M$ and $d(x, y)$ is called the distance from the point $x$ to the point $y$. The three distance properties are as follows: For all $x, y, z \in M$ we have
(a) positive definiteness: $d(x, y) \geq 0$, and $d(x, y)=0$ if and only if $x=y$.
(b) symmetry: $d(x, y)=d(y, x)$.
(c) triangle inequality: $d(x, z) \leq d(x, y)+d(y, z)$.

The function $d$ is also called the distance function. Strictly speaking, it is the pair $(M, d)$ which is a metric space, but we will follow the common practice of referring to "the metric space $M$," and leave to you the job of inferring the correct metric.

The main examples of metric spaces are $\mathbb{R}, \mathbb{R}^{m}$, and their subsets. The metric on $\mathbb{R}$ is $d(x, y)=|x-y|$ where $x, y \in \mathbb{R}$ and $|x-y|$ is the magnitude of $x-y$. The metric on $\mathbb{R}^{m}$ is the Euclidean length of $x-y$ where $x, y$ are vectors in $\mathbb{R}^{m}$. Namely,

$$
d(x, y)=\sqrt{\left(x_{1}-y_{1}\right)^{2}+\ldots+\left(x_{m}-y_{m}\right)^{2}}
$$

for $x=\left(x_{1}, \ldots, x_{m}\right)$ and $y=\left(y_{1}, \ldots, y_{m}\right)$.
Since Euclidean length satisfies the three distance properties, $d$ is a bona fide metric and it makes $\mathbb{R}^{m}$ into a metric space. A subset $M \subset \mathbb{R}^{m}$ becomes a metric space when we declare the distance between points of $M$ to be their Euclidean distance apart as points in $\mathbb{R}^{m}$. We say that $M$ inherits its metric from $\mathbb{R}^{m}$ and is a metric subspace of $\mathbb{R}^{m}$. Figure 27 shows a few subsets of $\mathbb{R}^{2}$ to suggest some interesting metric spaces.

There is also one metric that is hard to picture but valuable as a source for counterexamples, the discrete metric. Given any set $M$, define the distance between distinct points of $M$ to be 1 and the distance between every point and itself to be 0 . This is a metric. See Exercise 4. If $M$ consists of three points, say $M=\{a, b, c\}$, you can think of the vertices of the unit equilateral triangle as a model for $M$. See Figure 28. They have mutual distance 1 from each other. If $M$ consists of one, two, or four points can you think of a model for the discrete metric on $M$ ? More challenging is to imagine the discrete metric on $\mathbb{R}$. All points, by definition of the discrete metric, lie at unit distance from each other.

## Convergent Sequences and Subsequences

A sequence of points in a metric space $M$ is a list $p_{1}, p_{2}, \ldots$ where the points $p_{n}$ belong to $M$. Repetition is allowed, and not all the points of $M$ need to appear in the list. Good notation for a sequence is $\left(p_{n}\right)$, or $\left(p_{n}\right)_{n \in \mathbb{N}}$. The notation $\left\{p_{n}\right\}$ is also used but it is too easily confused with the set of points making up the sequence. The difference between $\left(p_{n}\right)_{n \in \mathbb{N}}$ and $\left\{p_{n}: n \in \mathbb{N}\right\}$ is that in the former case


Figure 27 Five metric spaces - a closed outward spiral, a Hawaiian earring, a topologist's sine circle, an infinite television antenna, and Zeno's maze


Figure 28 The vertices of the unit equilateral triangle form a discrete metric space.
the sequence prescribes an ordering of the points, while in the latter the points get jumbled together. For example, the sequences $1,2,3, \ldots$ and $1,2,1,3,2,1,4,3,2,1, \ldots$ are different sequences but give the same set of points, namely $\mathbb{N}$.

Formally, a sequence in $M$ is a function $f: \mathbb{N} \rightarrow M$. The $n^{\text {th }}$ term in the sequence is $f(n)=p_{n}$. Clearly, every sequence defines a function $f: \mathbb{N} \rightarrow M$ and conversely, every function $f: \mathbb{N} \rightarrow M$ defines a sequence in $M$. The sequence $\left(p_{n}\right)$ converges to the limit $p$ in $M$ if

$$
\begin{aligned}
& \forall \epsilon>0 \exists N \in \mathbb{N} \text { such that } \\
& n \in \mathbb{N} \text { and } n \geq N \quad \Rightarrow \quad d\left(p_{n}, p\right)<\epsilon
\end{aligned}
$$

Limits are unique in the sense that if $\left(p_{n}\right)$ converges to $p$ and if $\left(p_{n}\right)$ also converges to $p^{\prime}$ then $p=p^{\prime}$. The reason is this. Given any $\epsilon>0$, there are integers $N$ and $N^{\prime}$ such that if $n \geq N$ then $d\left(p_{n}, p\right)<\epsilon$, while if $n \geq N^{\prime}$ then $d\left(p_{n}, p^{\prime}\right)<\epsilon$. Then for all $n \geq \max \left\{N, N^{\prime}\right\}$ we have

$$
d\left(p, p^{\prime}\right) \leq d\left(p, p_{n}\right)+d\left(p_{n}, p^{\prime}\right)<\epsilon+\epsilon=2 \epsilon .
$$

But $\epsilon$ is arbitrary and so $d\left(p, p^{\prime}\right)=0$ and $p=p^{\prime}$. (This is the $\epsilon$-principle again.)
We write $p_{n} \rightarrow p$, or $p_{n} \rightarrow p$ as $n \rightarrow \infty$, or

$$
\lim _{n \rightarrow \infty} p_{n}=p
$$

to indicate convergence. For example, in $\mathbb{R}$ the sequence $p_{n}=1 / n$ converges to 0 as $n \rightarrow \infty$. In $\mathbb{R}^{2}$ the sequence $(1 / n, \sin n)$ does not converge as $n \rightarrow \infty$. In the metric space $\mathbb{Q}$ (with the metric it inherits from $\mathbb{R}$ ) the sequence $1,1.4,1.414,1.4142, \ldots$ does not converge.

Just as a set can have a subset, a sequence can have a subsequence. For example, the sequence $2,4,6,8, \ldots$ is a subsequence of $1,2,3,4, \ldots$ The sequence $3,5,7,11,13,17, \ldots$ is a subsequence of $1,3,5,7,9, \ldots$, which in turn is a subsequence of $1,2,3,4, \ldots$ In general, if $\left(p_{n}\right)_{n \in \mathbb{N}}$ and $\left(q_{k}\right)_{k \in \mathbb{N}}$ are sequences and if there is a sequence $n_{1}<n_{2}<n_{3}<\ldots$ of positive integers such that for each $k \in \mathbb{N}$ we have $q_{k}=p_{n_{k}}$ then $\left(q_{k}\right)$ is a subsequence of $\left(p_{n}\right)$. Note that the terms in the subsequence occur in the same order as in the mother sequence.

1 Theorem Every subsequence of a convergent sequence in $M$ converges and it converges to the same limit as does the mother sequence.

Proof Let $\left(q_{k}\right)$ be a subsequence of $\left(p_{n}\right), q_{k}=p_{n_{k}}$, where $n_{1}<n_{2}<\ldots$. Assume that $\left(p_{n}\right)$ converges to $p$ in $M$. Given $\epsilon>0$, there is an $N$ such that for all $n \geq N$ we have $d\left(p_{n}, p\right)<\epsilon$. Since $n_{1}, n_{2}, \ldots$ are positive integers we have $k \leq n_{k}$ for all $k$. Thus, if $k \geq N$ then $n_{k} \geq N$ and $d\left(q_{k}, p\right)<\epsilon$. Hence $\left(q_{k}\right)$ converges to $p$.

A common way to state Theorem 1 is that limits are unaffected when we pass to a subsequence.

## 2 Continuity

In linear algebra the objects of interest are linear transformations. In real analysis the objects of interest are functions, especially continuous functions. A function $f$ from the metric space $M$ to the metric space $N$ is just that; $f: M \rightarrow N$ and $f$ sends points $p \in M$ to points $f p \in N$. The function maps $M$ to $N$. The way you should think of functions - as devices, not formulas - is discussed in Section 4 of Chapter 1. The most common type of function maps $M$ to $\mathbb{R}$. It is a real-valued function of the variable $p \in M$.

Definition A function $f: M \rightarrow N$ is continuous if it preserves sequential convergence: $f$ sends convergent sequences in $M$ to convergent sequences in $N$, limits being sent to limits. That is, for each sequence $\left(p_{n}\right)$ in $M$ which converges to a limit $p$ in $M$, the image sequence $\left(f\left(p_{n}\right)\right)$ converges to $f p$ in $N$.

Here and in what follows, the notation $f p$ is often used as convenient shorthand for $f(p)$. The metrics on $M$ and $N$ are $d_{M}$ and $d_{N}$. We will often refer to either metric as just $d$.

2 Theorem The composite of continuous functions is continuous.
Proof Let $f: M \rightarrow N$ and $g: N \rightarrow P$ be continuous and assume that

$$
\lim _{n \rightarrow \infty} p_{n}=p
$$

in $M$. Since $f$ is continuous, $\lim _{n \rightarrow \infty} f\left(p_{n}\right)=f p$. Since $g$ is continuous, $\lim _{n \rightarrow \infty} g\left(f\left(p_{n}\right)\right)=$ $g(f p)$ and therefore $g \circ f: M \rightarrow P$ is continuous. See Figure 29 on the next page.

Moral The sequence condition is the easy way to tell at a glance whether a function is continuous.


Figure 29 The composite function $g \circ f$

There are two "obviously" continuous functions.
3 Proposition For every metric space $M$ the identity map id : $M \rightarrow M$ is continuous, and so is every constant function $f: M \rightarrow N$.

Proof Let $p_{n} \rightarrow p$ in $M$. Then $\operatorname{id}\left(p_{n}\right)=p_{n} \rightarrow p=\operatorname{id}(p)$ as $n \rightarrow \infty$ which gives continuity of the identity map. Likewise, if $f(x)=q \in N$ for all $x \in M$ and if $p_{n} \rightarrow p$ in $M$ then $f p=q$ and $f\left(p_{n}\right)=q$ for all $n$. Thus $f\left(p_{n}\right) \rightarrow f p$ as $n \rightarrow \infty$ which gives continuity of the constant function $f$.

## Homeomorphism

Vector spaces are isomorphic if there is a linear bijection from one to the other. When are metric spaces isomorphic? They should "look the same." The letters Y and T look the same; and they look different from the letter O . If $f: M \rightarrow N$ is a bijection and $f$ is continuous and the inverse bijection $f^{-1}: N \rightarrow M$ is also continuous then $f$ is a homeomorphism ${ }^{\dagger}$ (or a "homeo" for short) and $M, N$ are homeomorphic. We write $M \cong N$ to indicate that $M$ and $N$ are homeomorphic. $\cong$ is an equivalence relation: $M \cong M$ since the identity map is a homeomorphism $M \rightarrow M ; M \cong N$ clearly implies that $N \cong M$; and the previous theorem shows that the composite of homeomorphisms is a homeomorphism.

Geometrically speaking, a homeomorphism is a bijection that can bend, twist, stretch, and wrinkle the space $M$ to make it coincide with $N$, but it cannot rip,

[^0]puncture, shred, or pulverize $M$ in the process. The basic questions to ask about metric spaces are:
(a) Given $M$ and $N$, are they homeomorphic?
(b) What are the continuous functions from $M$ to $N$ ?

A major goal of this chapter is to show you how to answer these questions in many cases. For example, is the circle homeomorphic to the interval? To the sphere? etc. Figure 30 indicates that the circle and the (perimeter of the) triangle are homeomorphic, while Figure 15 shows that $(a, b)$, the semicircle, and $\mathbb{R}$ are homeomorphic.


Figure 30 The circle and triangle are homeomorphic.
A natural question that should occur to you is whether continuity of $f^{-1}$ is actually implied by continuity of a bijection $f$. It is not. Here is an instructive example.

Consider the interval $[0,2 \pi)=\{x \in \mathbb{R}: 0 \leq x<2 \pi\}$ and define $f:[0,2 \pi) \rightarrow S^{1}$ to be the mapping $f(x)=(\cos x, \sin x)$ where $S^{1}$ is the unit circle in the plane. The mapping $f$ is a continuous bijection, but the inverse bijection is not continuous. For there is a sequence of points $\left(z_{n}\right)$ on $S^{1}$ in the fourth quadrant that converges to $p=(1,0)$ from below, and $f^{-1}\left(z_{n}\right)$ does not converge to $f^{-1}(p)=0$. Rather it converges to $2 \pi$. Thus, $f$ is a continuous bijection whose inverse bijection fails to be continuous. See Figure 31. In Exercises 49 and 50 you are asked to find worse examples of continuous bijections that are not homeomorphisms.

To build your intuition about continuous mappings and homeomorphisms, consider the following examples shown in Figure 32 - the unit circle in the plane, a trefoil knot in $\mathbb{R}^{3}$, the perimeter of a square, the surface of a donut (the 2-torus), the surface


Figure $31 f$ wraps $[0,2 \pi)$ bijectively onto the circle.
of a ceramic coffee cup, the unit interval $[0,1]$, the unit disc including its boundary. Equip all with the inherited metric. Which should be homeomorphic to which?


Figure 32 Seven metric spaces

## The ( $\epsilon, \delta$ )-Condition

The following theorem presents the more familiar (but equivalent!) definition of continuity using $\epsilon$ and $\delta$. It corresponds to the definition given in Chapter 1 for real-valued functions of a real variable.

4 Theorem $f: M \rightarrow N$ is continuous if and only if it satisfies the $(\boldsymbol{\epsilon}, \boldsymbol{\delta})$-condition: For each $\epsilon>0$ and each $p \in M$ there exists $\delta>0$ such that if $x \in M$ and $d_{M}(x, p)<\delta$ then $d_{N}(f x, f p)<\epsilon$.

Proof Suppose that $f$ is continuous. It preserves sequential convergence. From the supposition that $f$ fails to satisfy the $(\epsilon, \delta)$-condition at some $p \in M$ we will derive a contradiction. If the $(\epsilon, \delta)$-condition fails at $p$ then there exists $\epsilon>0$ such that for each $\delta>0$ there is a point $x$ with $d(x, p)<\delta$ and $d(f x, f p) \geq \epsilon$. Taking $\delta=1 / n$ we get a sequence $\left(x_{n}\right)$ with $d\left(x_{n}, p\right)<1 / n$ and $d\left(f\left(x_{n}\right), f p\right) \geq \epsilon$, which contradicts preservation of sequential convergence. For $x_{n} \rightarrow p$ but $f\left(x_{n}\right)$ does not converge to $f p$, which contradicts the fact that $f$ preserves sequential convergence. Since the supposition that $f$ fails to satisfy the $(\epsilon, \delta)$-condition leads to a contradiction, $f$ actually does satisfy the $(\epsilon, \delta)$-condition.

To check the converse, suppose that $f$ satisfies the $(\epsilon, \delta)$-condition at $p$. For each sequence $\left(x_{n}\right)$ in $M$ that converges to $p$ we must show $f\left(x_{n}\right) \rightarrow f p$ in $N$ as $n \rightarrow \infty$. Let $\epsilon>0$ be given. There is $\delta>0$ such that $d_{M}(x, p)<\delta \Rightarrow d_{N}(f x, f p)<\epsilon$. Convergence of $x_{n}$ to $p$ implies there is an integer $K$ such that for all $n \geq K$ we have $d_{M}\left(x_{n}, p\right)<\delta$, and therefore $d_{N}\left(f\left(x_{n}\right), f p\right)<\epsilon$. That is, $f\left(x_{n}\right) \rightarrow f p$ as $n \rightarrow \infty$. See also Exercise 13.

## 3 The Topology of a Metric Space

Now we come to two basic concepts in a metric space theory - closedness and openness. Let $M$ be a metric space and let $S$ be a subset of $M$. A point $p \in M$ is a limit of $S$ if there exists a sequence $\left(p_{n}\right)$ in $S$ that converges to it. ${ }^{\dagger}$

[^1]Definition $S$ is a closed set if it contains all its limits. ${ }^{\dagger}$
Definition $S$ is an open set if for each $p \in S$ there exists an $r>0$ such that

$$
d(p, q)<r \quad \Rightarrow \quad q \in S
$$

5 Theorem Openness is dual to closedness: The complement of an open set is a closed set and the complement of a closed set is an open set.

Proof Suppose that $S \subset M$ is an open set. We claim that $S^{c}$ is a closed set. If $p_{n} \rightarrow p$ and $p_{n} \in S^{c}$ we must show that $p \in S^{c}$. Well, if $p \notin S^{c}$ then $p \in S$ and, since $S$ is open, there is an $r>0$ such that

$$
d(p, q)<r \quad \Rightarrow \quad q \in S
$$

Since $p_{n} \rightarrow p$, we have $d\left(p, p_{n}\right)<r$ for all large $n$, which implies that $p_{n} \in S$, contrary to the sequence being in $S^{c}$. Since the supposition that $p$ lies in $S$ leads to a contradiction, $p$ actually does lie in $S^{c}$, proving that $S^{c}$ is a closed set.

Suppose that $S$ is a closed set. We claim that $S^{c}$ is open. Take any $p \in S^{c}$. If there fails to exist an $r>0$ such that

$$
d(p, q)<r \quad \Rightarrow \quad q \in S^{c}
$$

then for each $r=1 / n$ with $n=1,2, \ldots$ there exists a point $p_{n} \in S$ such that $d\left(p, p_{n}\right)<1 / n$. This sequence in $S$ converges to $p \in S^{c}$, contrary to closedness of $S$. Therefore there actually does exist an $r>0$ such that

$$
d(p, q)<r \quad \Rightarrow \quad q \in S^{c}
$$

which proves that $S^{c}$ is an open set.

Most sets, like doors, are neither open nor closed, but ajar. Keep this in mind. For example neither $(a, b]$ nor its complement is closed in $\mathbb{R} ;(a, b]$ is neither closed nor open. Unlike doors, however, sets can be both open and closed at the same time. For example, the empty set $\emptyset$ is a subset of every metric space and it is always closed. There are no sequences and no limits to even worry about. Similarly the full metric space $M$ is a closed subset of itself: For it certainly contains the limit of

[^2]every sequence that converges in $M$. Thus, $\emptyset$ and $M$ are closed subsets of $M$. Their complements, $M$ and $\emptyset$, are therefore open: $\emptyset$ and $M$ are both closed and open.

Subsets of $M$ that are both closed and open are clopen. See also Exercise 125. It turns out that in $\mathbb{R}$ the only clopen sets are $\emptyset$ and $\mathbb{R}$. In $\mathbb{Q}$, however, things are quite different, sets such as $\{r \in \mathbb{Q}:-\sqrt{2}<r<\sqrt{2}\}$ being clopen in $\mathbb{Q}$. To summarize,

> A subset of a metric space can be
> closed, open, both, or neither.

You should expect the "typical" subset of a metric space to be neither closed nor open.

The topology of $M$ is the collection $\mathcal{T}$ of all open subsets of $M$.
6 Theorem $\mathcal{T}$ has three properties: ${ }^{\dagger}$ as a system it is closed under union, finite intersection, and it contains $\emptyset, M$. That is,
(a) Every union of open sets is an open set.
(b) The intersection of finitely many open sets is an open set.
(c) $\emptyset$ and $M$ are open sets.

Proof (a) If $\left\{U_{\alpha}\right\}$ is any collection ${ }^{\ddagger}$ of open subsets of $M$ and $V=\mathbf{U} U_{\alpha}$ then $V$ is open. For if $p \in V$ then $p$ belongs to at least one $U_{\alpha}$ and there is an $r>0$ such that

$$
d(p, q)<r \quad \Rightarrow \quad q \in U_{\alpha} .
$$

Since $U_{\alpha} \subset V$, this implies that all such $q$ lie in $V$, proving that $V$ is open.
(b) If $U_{1}, \ldots, U_{n}$ are open sets and $W=\bigcap U_{k}$ then $W$ is open. For if $p \in W$ then for each $k, 1 \leq k \leq n$, then there is an $r_{k}>0$ such that

$$
d(p, q)<r_{k} \quad \Rightarrow \quad q \in U_{k}
$$

Take $r=\min \left\{r_{1}, \ldots, r_{n}\right\}$. Then $r>0$ and

$$
d(p, q)<r \quad \Rightarrow \quad q \in U_{k},
$$

[^3]for each $k$; i.e., $q \in W=\bigcap U_{k}$, proving that $W$ is open.
(c) It is clear that $\emptyset$ and $M$ are open sets.

7 Corollary The intersection of any number of closed sets is a closed set; the finite union of closed sets is a closed set; $\emptyset$ and $M$ are closed sets.

Proof Take complements and use De Morgan's laws. If $\left\{K_{\alpha}\right\}$ is a collection of closed sets then $U_{\alpha}=\left(K_{\alpha}\right)^{c}$ is open and

$$
K=\bigcap K_{\alpha}=\left(\mathbf{U} U_{\alpha}\right)^{c}
$$

Since $\mathbf{U} U_{\alpha}$ is open, its complement $K$ is closed. Similarly, a finite union of closed sets is the complement of the finite intersection of their complements, and is a closed set.

What about an infinite union of closed sets? Generally, it is not closed. For example, the interval $[1 / n, 1]$ is closed in $\mathbb{R}$, but the union of these intervals as $n$ ranges over $\mathbb{N}$ is the interval $(0,1]$ which is not closed in $\mathbb{R}$. Neither is the infinite intersection of open sets open in general.

Two sets whose closedness/openness properties are basic are:

$$
\begin{aligned}
\lim S & =\{p \in M: p \text { is a limit of } S\} \\
M_{r} p & =\{q \in M: d(p, q)<r\}
\end{aligned}
$$

The former is the limit set of $S$; the latter is the $\boldsymbol{r}$-neighborhood of $p$.
8 Theorem $\lim S$ is a closed set and $M_{r} p$ is an open set.
Proof Simple but not immediate! See Figure 33.
Suppose that $p_{n} \rightarrow p$ and each $p_{n}$ lies in $\lim S$. We claim that $p \in \lim S$. Since $p_{n}$ is a limit of $S$ there is a sequence $\left(p_{n, k}\right)_{k \in \mathbb{N}}$ in $S$ that converges to $p_{n}$ as $k \rightarrow \infty$. Thus there exists $q_{n}=p_{n, k(n)} \in S$ such that

$$
d\left(p_{n}, q_{n}\right)<\frac{1}{n}
$$

Then, as $n \rightarrow \infty$ we have

$$
d\left(p, q_{n}\right) \leq d\left(p, p_{n}\right)+d\left(p_{n}, q_{n}\right) \rightarrow 0
$$

which implies that $q_{n} \rightarrow p$, so $p \in \lim S$, which completes the proof that $\lim S$ is a closed set.


Figure $33 S=(0,1) \times(0,1)$ and $p_{n}=(1 / n, 0)$ converges to $p=(0,0)$ as $n \rightarrow \infty$. Each $p_{n}$ is the limit of the sequence $p_{n, k}=(1 / n, 1 / k)$ as $k \rightarrow \infty$. The sequence $q_{n}=(1 / n, 1 / n)$ lies in $S$ and converges to $(0,0)$. Hence: The limits of limits are limits.


Figure 34 Why the $r$-neighborhood of $p$ is an open set

To check that $M_{r} p$ is an open set, take any $q \in M_{r} p$ and observe that

$$
s=r-d(p, q)>0
$$

By the triangle inequality, if $d(q, x)<s$ then

$$
d(p, x) \leq d(p, q)+d(q, x)<r
$$

and hence $M_{s} q \subset M_{r} p$. See Figure 34. Since each $q \in M_{r} p$ has some $M_{s} q$ that is contained in $M_{r} p, M_{r} p$ is an open set.

9 Corollary The interval $(a, b)$ is open in $\mathbb{R}$ and so are $(-\infty, b),(a, \infty)$, and $(-\infty, \infty)$. The interval $[a, b]$ is closed in $\mathbb{R}$.

Proof $(a, b)$ is the $r$-neighborhood of its midpoint $m=(a+b) / 2$ where $r=(b-a) / 2$. Therefore $(a, b)$ is open in $\mathbb{R}$. Since the union of open sets is open we see that

$$
\bigcup_{n \in \mathbb{N}}(b-n, b-1 / n)=(-\infty, b)
$$

is open. The same applies to $(a, \infty)$. The whole metric space $\mathbb{R}=(-\infty, \infty)$ is always open in itself.

Since the complement of $[a, b]$ is the open set $(-\infty, a) \cup(b, \infty)$, the interval $[a, b]$ is closed.

10 Corollary $\lim S$ is the "smallest" closed set that contains $S$ in the sense that if $K \supset S$ and $K$ is closed then $K \supset \lim S$.

Proof Obvious. $K$ must contain the limit of each sequence in $K$ that converges in $M$ and therefore it must contain the limit of each sequence in $S \subset K$ that converges in $M$. These limits are exactly $\lim S$.

We refer to $\lim S$ as the closure of $S$ and denote it also as $\bar{S}$. You start with $S$ and make it closed by adding all its limits. You don't need to add any more points because limits of limits are limits. That is, $\lim (\lim S)=\lim S$. An operation like this is called idempotent. Doing the operation twice produces the same outcome as doing it once.

A neighborhood of a point $p$ in $M$ is any open set $V$ that contains $p$. Theorem 8 implies that $V=M_{r} p$ is a neighborhood of $p$. Eventually, you will run across the phrase "closed neighborhood" of $p$, which refers to a closed set that contains an open set that contains $p$. However, until further notice all neighborhoods are open.

Usually, sets defined by strict inequalities are open while those defined by equalities or nonstrict inequalities are closed. Examples of closed sets in $\mathbb{R}$ are finite sets, $[a, b], \mathbb{N}$, and the set $\{0\} \cup\{1 / n: n \in N\}$. Each contains all its limits. Examples of open sets in $\mathbb{R}$ are open intervals, bounded or not.

## Topological Description of Continuity

A property of a metric space or of a mapping between metric spaces that can be described solely in terms of open sets (or equivalently, in terms of closed sets) is called a topological property. The next result describes continuity topologically.


Figure 35 The function $f:(x, y) \mapsto x^{2}+y^{2}+2$ and its graph over the preimage of $[3,6]$

Let $f: M \rightarrow N$ be given. The preimage ${ }^{\dagger}$ of a set $V \subset N$ is

$$
f^{\mathrm{pre}}(V)=\{p \in M: f(p) \in V\}
$$

For example, if $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ is the function defined by the formula

$$
f(x, y)=x^{2}+y^{2}+2
$$

then the preimage of the interval $[3,6]$ in $\mathbb{R}$ is the annulus in the plane with inner radius 1 and outer radius 2. Figure 35 shows the domain of $f$ as $\mathbb{R}^{2}$ and the target

[^4]as $\mathbb{R}$. The range is the set of real numbers $\geq 2$. The graph of $f$ is a paraboloid with lowest point $(0,0,2)$. The second part of the figure shows the portion of the graph lying above the annulus. You will find it useful to keep in mind the distinctions among the concepts: function, range, and graph.

11 Theorem The following are equivalent for continuity of $f: M \rightarrow N$.
(i) The $(\epsilon, \delta)$-condition.
(ii) The sequential convergence preservation condition.
(iii) The closed set condition: The preimage of each closed set in $N$ is closed in M.
(iv) The open set condition: The preimage of each open set in $N$ is open in $M$.

Proof Totally natural! By Theorem 4, (i) implies (ii).
(ii) implies (iii). If $K \subset N$ is closed in $N$ and $p_{n} \in f^{\text {pre }}(K)$ converges to $p \in M$ then we claim that $p \in f^{\text {pre }}(K)$. By (ii), $f$ preserves sequential convergence so

$$
\lim _{n \rightarrow \infty} f\left(p_{n}\right)=f p
$$

Since $K$ is closed in $N, f p \in K$, so $p \in f^{\text {pre }}(K)$, and we see that $f^{\text {pre }}(K)$ is closed in $M$. Thus (ii) implies (iii).
(iii) implies (iv). This follows by taking complements: $\left(f^{\text {pre }}(U)\right)^{c}=f^{\text {pre }}\left(U^{c}\right)$.
(iv) implies (i). Let $\epsilon>0$ and $p \in M$ be given. $N_{\epsilon}(f p)$ is open in $N$, so its preimage $U=f^{\text {pre }}\left(N_{\epsilon}(f p)\right)$ is open in $M$. The point $p$ belongs to the preimage so openness of $U$ implies there is a $\delta>0$ such that $M_{\delta}(p) \subset U$. Then

$$
f\left(M_{\delta}(p)\right) \subset f U \subset N_{\epsilon}(f p)
$$

gives the $\epsilon, \delta$ condition, $d_{M}(p, x)<\delta \Rightarrow d_{N}(f p, f x)<\epsilon$. See Figure 36 .

I hope you find the closed and open set characterizations of continuity elegant. Note that no explicit mention is made of the metric. The open set condition is purely topological. It would be perfectly valid to take as a definition of continuity that the preimage of each open set is open. In fact this is exactly what's done in general topology.

12 Corollary $A$ homeomorphism $f: M \rightarrow N$ bijects the collection of open sets in $M$ to the collection of open sets in $N$. It bijects the topologies.


Figure 36 The $\epsilon, \delta$ - condition for a continuous function $f: M \rightarrow N$

Proof Let $V$ be an open set in $N$. By Theorem 11, since $f$ is continuous, the preimage of $V$ is open in $M$. Since $f$ is a bijection, this preimage $U=\{p \in M: f p \in V\}$ is exactly the image of $V$ by the inverse bijection, $U=f^{-1}(V)$. The same thing can be said about $f^{-1}$ since $f^{-1}$ is also a homeomorphism. That is, $V=f U$. Thus, sending $U$ to $f U$ bijects the topology of $M$ to the topology of $N$.

Because of this corollary, a homeomorphism is also called a topological equivalence.

In general, continuous maps do not need to send open sets to open sets. For example, the squaring map $x \mapsto x^{2}$ from $\mathbb{R}$ to $\mathbb{R}$ is continuous but it sends the open interval $(-1,1)$ to the nonopen interval $[0,1)$. See also Exercise 28.

## Inheritance

If a set $S$ is contained in a metric subspace $N \subset M$ you need to be careful when you say that $S$ is open or closed. For example,

$$
S=\{x \in \mathbb{Q}:-\pi<x<\pi\}
$$

is a subset of the metric subspace $\mathbb{Q} \subset \mathbb{R}$. It is both open and closed with respect to $\mathbb{Q}$ but is neither open nor closed with respect to $\mathbb{R}$. To avoid this kind of ambiguity it is best to say that $S$ is clopen "with respect to $\mathbb{Q}$ but not with respect to $\mathbb{R}$," or more briefly that $S$ is clopen "in $\mathbb{Q}$ but not in $\mathbb{R}$." Nevertheless there is a simple relation between the topologies of $M$ and $N$ when $N$ is a metric subspace of $M$.

13 Inheritance Principle Every metric subspace $N$ of $M$ inherits its topology from $M$ in the sense that each subset $V \subset N$ which is open in $N$ is actually the intersection $V=N \cap U$ for some $U \subset M$ that is open in $M$, and vice versa.

Proof It all boils down to the fact that for each $p \in N$ we have

$$
N_{r} p=N \cap M_{r} p
$$

After all, $N_{r} p$ is the set of $x \in N$ such that $d_{N}(x, p)<r$ and this is exactly the same as the set of those $x \in M_{r} p$ that belong to $N$. Therefore $N$ inherits its $r$ neighborhoods from $M$. Since its open sets are unions of its $r$-neighborhoods, $N$ also inherits its open sets from $M$.

In more detail, if $V$ is open in $N$ then it is the union of those $N_{r} p$ with $N_{r} p \subset V$. Each such $N_{r} p$ is $N \cap M_{r} p$ and the union of these $M_{r} p$ is $U$, an open subset of $M$. The intersection $N \cap U$ equals $V$. Conversely, if $U$ is any open subset of $M$ and $p \in V=N \cap U$ then openness of $U$ implies there is an $M_{r} p \subset U$. Thus $N_{r} p=N \cap M_{r} p \subset V$, which shows that $V$ is open in $N$.

14 Corollary Every metric subspace of $M$ inherits its closed sets from $M$.

Proof By "inheriting its closed sets" we mean that each closed subset $L \subset N$ is the intersection $N \cap K$ for some closed subset $K \subset M$ and vice versa. The proof consists of two words: "Take complements." See also Exercise 34.

Let's return to the example with $\mathbb{Q} \subset \mathbb{R}$ and $S=\{x \in \mathbb{Q}:-\pi<x<\pi\}$. The set $S$ is clopen in $\mathbb{Q}$, so we should have $S=\mathbb{Q} \cap U$ for some open set $U \subset \mathbb{R}$ and $S=\mathbb{Q} \cap K$ for some closed set $K \subset \mathbb{R}$. In fact $U$ and $K$ are

$$
U=(-\pi, \pi) \quad \text { and } \quad K=[-\pi, \pi] .
$$

15 Corollary Assume that $N$ is a metric subspace of $M$ and also is a closed subset of $M$. A set $L \subset N$ is closed in $N$ if and only if it is closed in $M$. Similarly, if $N$ is a metric subspace of $M$ and also is an open subset of $M$ then $U \subset N$ is open in $N$ if and only if it is open in $M$.

Proof The proof is left to the reader as Exercise 34.

## Product Metrics

We next define a metric on the Cartesian product $M=X \times Y$ of two metric spaces. There are three natural ways to do so:

$$
\begin{aligned}
d_{E}\left(p, p^{\prime}\right) & =\sqrt{d_{X}\left(x, x^{\prime}\right)^{2}+d_{Y}\left(y, y^{\prime}\right)^{2}} \\
d_{\max }\left(p, p^{\prime}\right) & =\max \left\{d_{X}\left(x, x^{\prime}\right), d_{Y}\left(y, y^{\prime}\right)\right\} \\
d_{\text {sum }}\left(p, p^{\prime}\right) & =d_{X}\left(x, x^{\prime}\right)+d_{Y}\left(y, y^{\prime}\right)
\end{aligned}
$$

where $p=(x, y)$ and $p^{\prime}=\left(x^{\prime}, y^{\prime}\right)$ belong to $M$. $\left(d_{E}\right.$ is the Euclidean product metric.) The proof that these expressions actually define metrics on $M$ is left as Exercise 38.

16 Proposition $d_{\max } \leq d_{E} \leq d_{\text {sum }} \leq 2 d_{\max }$.
Proof Dropping the smaller term inside the square root shows that $d_{\max } \leq d_{E}$; comparing the square of $d_{E}$ and the square of $d_{\text {sum }}$ shows that the latter has the terms of the former and the cross term besides, so $d_{E} \leq d_{\text {sum }}$; and clearly $d_{\text {sum }}$ is no larger than twice its greater term, so $d_{\text {sum }} \leq 2 d_{\text {max }}$.

17 Convergence in a Product Space The following are equivalent for a sequence $p_{n}=\left(p_{1 n}, p_{2 n}\right)$ in $M=M_{1} \times M_{2}$ :
(a) $\left(p_{n}\right)$ converges with respect to the metric $d_{\max }$.
(b) $\left(p_{n}\right)$ converges with respect to the metric $d_{E}$.
(c) $\left(p_{n}\right)$ converges with respect to the metric $d_{\text {sum }}$.
(d) ( $p_{1 n}$ ) and ( $p_{2 n}$ ) converge in $M_{1}$ and $M_{2}$ respectively.

Proof This is immediate from Proposition 16.
18 Corollary If $f: M \rightarrow N$ and $g: X \rightarrow Y$ are continuous then so is their Cartesian product $f \times g$ which sends $(p, x) \in M \times X$ to $(f p, g x) \in N \times Y$.

Proof If $\left(p_{n}, x_{n}\right) \rightarrow(p, x)$ in $M \times X$ then Theorem 17 implies $p_{n} \rightarrow p$ in $M$ and $x_{n} \rightarrow x$ in $X$. By continuity, $f\left(p_{n}\right) \rightarrow f p$ and $g\left(x_{n}\right) \rightarrow g x$. By Theorem 17, $\left(f\left(p_{n}\right), g\left(x_{n}\right)\right) \rightarrow(f p, g x)$ which gives continuity of $f \times g$.

The three metrics make sense in the obvious way for a Cartesian product of $m \geq 3$ metric spaces. The inequality

$$
d_{\max } \leq d_{E} \leq d_{\mathrm{sum}} \leq m d_{\max }
$$

is proved in the same way, and we see that Theorem 17 holds also for the product of $m$ metric spaces. This gives

19 Corollary (Convergence in $\mathbb{R}^{m}$ ) A sequence of vectors $\left(v_{n}\right)$ in $\mathbb{R}^{m}$ converges in $\mathbb{R}^{m}$ if and only if each of its component sequences $\left(v_{i n}\right)$ converges, $1 \leq i \leq m$. The limit of the vector sequence is the vector

$$
v=\lim _{n \rightarrow \infty} v_{n}=\left(\lim _{n \rightarrow \infty} v_{1 n}, \lim _{n \rightarrow \infty} v_{2 n}, \ldots, \lim _{n \rightarrow \infty} v_{m n}\right) .
$$

The distance function $d: M \times M \rightarrow \mathbb{R}$ is a function from the metric space $M \times M$ to the metric space $\mathbb{R}$, so the following assertion makes sense.

20 Theorem $d$ is continuous.

Proof We check $(\epsilon, \delta)$-continuity with respect to the metric $d_{\text {sum }}$. Given $\epsilon>0$ we take $\delta=\epsilon$. If $d_{\text {sum }}\left((p, q),\left(p^{\prime}, q^{\prime}\right)\right)<\delta$ then the triangle inequality gives

$$
\begin{aligned}
d(p, q) & \leq d\left(p, p^{\prime}\right)+d\left(p^{\prime}, q^{\prime}\right)+d\left(q^{\prime}, q\right)<d\left(p^{\prime}, q^{\prime}\right)+\epsilon \\
d\left(p^{\prime}, q^{\prime}\right) & \leq d\left(p^{\prime}, p\right)+d(p, q)+d\left(q, q^{\prime}\right)<d(p, q)+\epsilon
\end{aligned}
$$

which gives

$$
d(p, q)-\epsilon<d\left(p^{\prime}, q^{\prime}\right)<d(p, q)+\epsilon
$$

Thus $\left|d\left(p^{\prime}, q^{\prime}\right)-d(p, q)\right|<\epsilon$ and we get continuity with respect to the metric $d_{\text {sum }}$. By Theorem 17 it does not matter which metric we use on $\mathbb{R} \times \mathbb{R}$.

As you can see, the use of $d_{\text {sum }}$ simplifies the proof by avoiding square root manipulations. The sum metric is also called the Manhattan metric or the taxicab metric. Figure 37 shows the "unit discs" with respect to these metrics in $\mathbb{R}^{2}$. See also Exercise 2.

21 Corollary The metrics $d_{\max }, d_{E}$, and $d_{\text {sum }}$ are continuous.

Proof Theorem 20 asserts that all metrics are continuous.

22 Corollary The absolute value is a continuous mapping $\mathbb{R} \rightarrow \mathbb{R}$. In fact the norm is a continuous mapping from any normed space to $\mathbb{R}$.

Proof $\|v\|=d(v, 0)$.


Figure 37 The unit disc in the max metric is a square, and in the sum metric it is a rhombus.

## Completeness

In Chapter 1 we discussed the Cauchy criterion for convergence of a sequence of real numbers. There is a natural way to carry these ideas over to a metric space $M$. The sequence $\left(p_{n}\right)$ in $M$ satisfies a Cauchy condition provided that for each $\epsilon>0$ there is an integer $N$ such that for all $k, n \geq N$ we have $d\left(p_{k}, p_{n}\right)<\epsilon$, and $\left(p_{n}\right)$ is said to be a Cauchy sequence. In symbols,

$$
\forall \epsilon>0 \exists N \text { such that } k, n \geq N \Rightarrow d\left(p_{k}, p_{n}\right)<\epsilon
$$

The terms of a Cauchy sequence "bunch together" as $n \rightarrow \infty$. Each convergent sequence $\left(p_{n}\right)$ is Cauchy. For if $\left(p_{n}\right)$ converges to $p$ as $n \rightarrow \infty$ then, given $\epsilon>0$, there is an $N$ such that for all $n \geq N$ we have

$$
d\left(p_{n}, p\right)<\frac{\epsilon}{2}
$$

By the triangle inequality, if $k, n \geq N$ then

$$
d\left(p_{k}, p_{n}\right) \leq d\left(p_{k}, p\right)+d\left(p, p_{n}\right)<\epsilon
$$

so convergence $\Rightarrow$ Cauchy.
Theorem 1.5 states that the converse is true in the metric space $\mathbb{R}$. Every Cauchy sequence in $\mathbb{R}$ converges to a limit in $\mathbb{R}$. In the general metric space, however, this
need not be true. For example, consider the metric space $\mathbb{Q}$ of rational numbers, equipped with the inherited metric $d(x, y)=|x-y|$, and consider the sequence

$$
\left(r_{n}\right)=(1.4,1.41,1.414,1.4142, \ldots)
$$

It is Cauchy. Given $\epsilon>0$, choose $N>-\log _{10} \epsilon$. If $k, n \geq N$ then $\left|r_{k}-r_{n}\right| \leq$ $10^{-N}<\epsilon$. Nevertheless, $\left(r_{n}\right)$ refuses to converge in $\mathbb{Q}$. After all, as a sequence in $\mathbb{R}$ it converges to $\sqrt{2}$, and if it also converges to some $r \in \mathbb{Q}$, then by uniqueness of limits in $\mathbb{R}$ we have $r=\sqrt{2}$, something we know is false. In brief, convergence $\Rightarrow$ Cauchy but not conversely.

A metric space $M$ is complete if each Cauchy sequence in $M$ converges to a limit in $M$. Theorem 1.5 states that $\mathbb{R}$ is complete.

23 Theorem $\mathbb{R}^{m}$ is complete.
Proof Let $\left(p_{n}\right)$ be a Cauchy sequence in $\mathbb{R}^{m}$. Express $p_{n}$ in components as

$$
p_{n}=\left(p_{1 n}, \ldots, p_{m n}\right)
$$

Because $\left(p_{n}\right)$ is Cauchy, each component sequence $\left(p_{i n}\right)_{n \in \mathbb{N}}$ is Cauchy. Completeness of $\mathbb{R}$ implies that the component sequences converge, and therefore the vector sequence converges.

24 Theorem Every closed subset of a complete metric space is a complete metric subspace.

Proof Let $A$ be a closed subset of the complete metric space $M$ and let $\left(p_{n}\right)$ be a Cauchy sequence in $A$ with respect to the inherited metric. It is of course also a Cauchy sequence in $M$ and therefore it converges to a limit $p$ in $M$. Since $A$ is closed we have $p \in A$.

25 Corollary Every closed subset of Euclidean space is a complete metric space.
Proof Obvious from the previous theorem and completeness of $\mathbb{R}^{m}$.

Remark Completeness is not a topological property. For example, consider $\mathbb{R}$ with its usual metric and $(-1,1)$ with the metric it inherits from $\mathbb{R}$. Although they are homeomorphic metric spaces, $\mathbb{R}$ is complete but $(-1,1)$ is not.

In Section 10 we explain how every metric space can be completed.

## 4 Compactness

Compactness is the single most important concept in real analysis. It is what reduces the infinite to the finite.

Definition A subset $A$ of a metric space $M$ is (sequentially) compact if every sequence $\left(a_{n}\right)$ in $A$ has a subsequence $\left(a_{n_{k}}\right)$ that converges to a limit in $A$.

The empty set and finite sets are trivial examples of compact sets. For a sequence $\left(a_{n}\right)$ contained in a finite set repeats a term infinitely often, and the corresponding constant subsequence converges.

Compactness is a good feature of a set. We will develop criteria to decide whether a set is compact. The first is the most often used, but beware! - its converse is generally false.

26 Theorem Every compact set is closed and bounded.
Proof Suppose that $A$ is a compact subset of the metric space $M$ and that $p$ is a limit of $A$. Does $p$ belong to $A$ ? There is a sequence $\left(a_{n}\right)$ in $A$ converging to $p$. By compactness, some subsequence $\left(a_{n_{k}}\right)$ converges to some $q \in A$, but every subsequence of a convergent sequence converges to the same limit as does the mother sequence, so $q=p$ and $p \in A$. Thus $A$ is closed.

To see that $A$ is bounded, choose and fix any point $p \in M$. Either $A$ is bounded or else for each $n \in \mathbb{N}$ there is a point $a_{n} \in A$ such that $d\left(p, a_{n}\right) \geq n$. Compactness implies that some subsequence $\left(a_{n_{k}}\right)$ converges. Convergent sequences are bounded, which contradicts the fact that $d\left(p, a_{n_{k}}\right) \rightarrow \infty$ as $k \rightarrow \infty$. Therefore $\left(a_{n}\right)$ cannot exist and for some large $r$ we have $A \subset M_{r} p$, which is what it means that $A$ is bounded. $\square$

27 Theorem The closed interval $[a, b] \subset \mathbb{R}$ is compact.
Proof Let $\left(x_{n}\right)$ be a sequence in $[a, b]$ and set

$$
C=\left\{x \in[a, b]: x_{n}<x \text { only finitely often }\right\} .
$$

Equivalently, for all but finitely many $n, x_{n} \geq x$. Since $a \in C$ we know that $C \neq \emptyset$. Clearly $b$ is an upper bound for $C$. By the least upper bound property of $\mathbb{R}$ there exists $c=$ l. u.b. $C$ with $c \in[a, b]$. We claim that a subsequence of $\left(x_{n}\right)$ converges to c. Suppose not, i.e., no subsequence of $\left(x_{n}\right)$ converges to $c$. Then for some $r>0, x_{n}$ lies in $(c-r, c+r)$ only finitely often, which implies that $c+r \in C$, contrary to $c$ being an upper bound for $C$. Hence some subsequence of $\left(x_{n}\right)$ does converge to $c$ and $[a, b]$ is compact.

To pass from $\mathbb{R}$ to $\mathbb{R}^{m}$ we think about compactness for Cartesian products.
28 Theorem The Cartesian product of two compact sets is compact.
Proof Let $\left(a_{n}, b_{n}\right) \in A \times B$ be given where $A \subset M$ and $B \subset N$ are compact. There exists a subsequence $\left(a_{n_{k}}\right)$ that converges to some point $a \in A$ as $k \rightarrow \infty$. The subsequence $\left(b_{n_{k}}\right)$ has a sub-subsequence $\left(b_{n_{k(\ell)}}\right)$ that converges to some $b \in B$ as $\ell \rightarrow \infty$. The sub-subsequence $\left(a_{n_{k(\ell)}}\right)$ continues to converge to the point $a$. Thus

$$
\left(a_{n_{k(\ell)}}, b_{n_{k(\ell)}}\right) \rightarrow(a, b)
$$

as $\ell \rightarrow \infty$. This implies that $A \times B$ is compact.
29 Corollary The Cartesian product of $m$ compact sets is compact.
Proof Write $A_{1} \times A_{2} \times \cdots \times A_{m}=A_{1} \times\left(A_{2} \times \cdots \times A_{m}\right)$ and perform induction on $m$. (Theorem 28 handles the bottom case $m=2$.)

30 Corollary Every box $\left[a_{1}, b_{1}\right] \times \cdots \times\left[a_{m}, b_{m}\right]$ in $\mathbb{R}^{m}$ is compact.
Proof Obvious from Theorem 27 and the previous corollary.

An equivalent formulation of these results is the
31 Bolzano-Weierstrass Theorem Every bounded sequence in $\mathbb{R}^{m}$ has a convergent subsequence.

Proof A bounded sequence is contained in a box, which is compact, and therefore the sequence has a subsequence that converges to a limit in the box. See also Exercise 11 .

Here is a simple fact about compacts.
32 Theorem Every closed subset of a compact is compact.
Proof If $A$ is a closed subset of the compact set $K$ and if $\left(a_{n}\right)$ is a sequence of points in $A$ then clearly $\left(a_{n}\right)$ is also a sequence of points in $K$, so by compactness of $K$ there is a subsequence $\left(a_{n_{k}}\right)$ converging to a limit $p \in K$. Since $A$ is closed, $p$ lies in $A$ which proves that $A$ is compact.

Now we come to the first partial converse to Theorem 26.

33 Heine-Borel Theorem Every closed and bounded subset of $\mathbb{R}^{m}$ is compact.
Proof Let $A \subset \mathbb{R}^{m}$ be closed and bounded. Boundedness implies that $A$ is contained in some box, which is compact. Since $A$ is closed, Theorem 32 implies that $A$ is compact. See also Exercise 11.

The Heine-Borel Theorem states that closed and bounded subsets of Euclidean space are compact, but it is vital ${ }^{\dagger}$ to remember that a closed and bounded subset of a general metric space may fail to be compact. For example, the set $\mathbb{N}$ of natural numbers equipped with the discrete metric is a complete metric space, it is closed in itself (as is every metric space), and it is bounded. But it is not compact. After all, what subsequence of $1,2,3, \ldots$ converges?

A more striking example occurs in the metric space $C([0,1], \mathbb{R})$ whose metric is $d(f, g)=\max \{|f(x)-g(x)|\}$. The space is complete but its closed unit ball is not compact. For example, the sequence of functions $f_{n}=x^{n}$ has no subsequence that converges with respect to the metric $d$. In fact every closed ball is noncompact.

## Ten Examples of Compact Sets

1. Any finite subset of a metric space, for instance the empty set.
2. Any closed subset of a compact set.

3 . The union of finitely many compact sets.
4. The Cartesian product of finitely many compact sets.
5. The intersection of arbitrarily many compact sets.

6 . The closed unit ball in $\mathbb{R}^{3}$.
7. The boundary of a compact set, for instance the unit 2 -sphere in $\mathbb{R}^{3}$.
8. The set $\{x \in \mathbb{R}: \exists n \in \mathbb{N}$ and $x=1 / n\} \cup\{0\}$.
9. The Hawaiian earring. See page 58.
10. The Cantor set. See Section 8.

## Nests of Compacts

If $A_{1} \supset A_{2} \supset \cdots \supset A_{n} \supset A_{n+1} \supset \ldots$ then $\left(A_{n}\right)$ is a nested sequence of sets. Its intersection is

$$
\bigcap_{n=1}^{\infty} A_{n}=\left\{p: \text { for each } n \text { we have } p \in A_{n}\right\}
$$

[^5]See Figure 38.


Figure 38 A nested sequence of sets
For example, we could take $A_{n}$ to be the disc $\left\{z \in \mathbb{R}^{2}:|z| \leq 1 / n\right\}$. The intersection of all the sets $A_{n}$ is then the singleton $\{0\}$. On the other hand, if $A_{n}$ is the ball $\left\{z \in \mathbb{R}^{3}:|z| \leq 1+1 / n\right\}$ then $\bigcap A_{n}$ is the closed unit ball $B^{3}$.

34 Theorem The intersection of a nested sequence of compact nonempty sets is compact and nonempty.

Proof Let $\left(A_{n}\right)$ be such a sequence. By Theorem 26, $A_{n}$ is closed. The intersection of closed sets is always closed. Thus, $\bigcap A_{n}$ is a closed subset of the compact set $A_{1}$ and is therefore compact. It remains to show that the intersection is nonempty.
$A_{n}$ is nonempty, so for each $n \in \mathbb{N}$ we can choose $a_{n} \in A_{n}$. The sequence ( $a_{n}$ ) lies in $A_{1}$ since the sets are nested. Compactness of $A_{1} \operatorname{implies}$ that $\left(a_{n}\right)$ has a subsequence $\left(a_{n_{k}}\right)$ converging to some point $p \in A_{1}$. The limit $p$ also lies in the set $A_{2}$ since except possibly for the first term, the subsequence $\left(a_{n_{k}}\right)$ lies in $A_{2}$ and $A_{2}$ is a closed set. The same is true for $A_{3}$ and for all the sets in the nested sequence. Thus, $p \in \bigcap A_{n}$ and $\bigcap A_{n}$ is shown to be nonempty.

The diameter of a nonempty set $S \subset M$ is the supremum of the distances $d(x, y)$ between points of $S$.

35 Corollary If in addition to being nested, nonempty, and compact, the sets $A_{n}$ have diameter that tends to 0 as $n \rightarrow \infty$ then $A=\bigcap A_{n}$ is a single point.

Proof For each $n \in \mathbb{N}, A$ is a subset of $A_{n}$, which implies that $A$ has diameter zero. Since distinct points lie at positive distance from each other, $A$ consists of at most one point, while by Theorem 34 it consists of at least one point. See also Exercise 52.


Figure 39 This nested sequence has empty intersection.
Figure 39 shows a nested sequence of nonempty noncompact sets with empty intersection. They are the open discs with center $(1 / n, 0)$ on the $x$-axis and radius $1 / n$. They contain no common point. (Their closures do intersect at a common point, the origin.)

## Continuity and Compactness

Next we discuss how compact sets behave under continuous transformations.
36 Theorem If $f: M \rightarrow N$ is continuous and $A$ is a compact subset of $M$ then $f A$ is a compact subset of $N$. That is, the continuous image of a compact is compact.

Proof Suppose that $\left(b_{n}\right)$ is a sequence in $f A$. For each $n \in \mathbb{N}$ choose a point $a_{n} \in A$ such that $f\left(a_{n}\right)=b_{n}$. By compactness of $A$ there exists a subsequence $\left(a_{n_{k}}\right)$ that converges to some point $p \in A$. By continuity of $f$ it follows that

$$
b_{n_{k}}=f\left(a_{n_{k}}\right) \rightarrow f p \in f A
$$

as $k \rightarrow \infty$. Thus, every sequence $\left(b_{n}\right)$ in $f A$ has a subsequence converging to a limit in $f A$, which shows that $f A$ is compact.

From Theorem 36 follows the natural generalization of the min/max theorem in Chapter 1 which concerns continuous real-valued functions defined on an interval $[a, b]$. See Theorem 1.23.

37 Corollary A continuous real-valued function defined on a compact set is bounded; it assumes maximum and minimum values.

Proof Let $f: M \rightarrow \mathbb{R}$ be continuous and let $A$ be a compact subset of $M$. Theorem 36 implies that $f A$ is a compact subset of $\mathbb{R}$, so by Theorem 26 it is closed and bounded. Thus, the greatest lower bound, $v$, and least upper bound, $V$, of $f A$ exist and belong to $f A$; i.e., there exist points $p, P \in A$ such that for all $a \in A$ we have $v=f p \leq f a \leq f P=V$.

## Homeomorphisms and Compactness

A homeomorphism is a bicontinuous bijection. Originally, compactness was called bicompactness. This is reflected in the next theorem.

38 Theorem If $M$ is compact and $M$ is homeomorphic to $N$ then $N$ is compact. Compactness is a topological property.

Proof $N$ is the continuous image of $M$, so by Theorem 36 it is compact.
39 Corollary $[0,1]$ and $\mathbb{R}$ are not homeomorphic.
Proof One is compact and the other isn't.
40 Theorem If $M$ is compact then a continuous bijection $f: M \rightarrow N$ is a homeomorphism - its inverse bijection $f^{-1}: N \rightarrow M$ is automatically continuous.

Proof Suppose that $q_{n} \rightarrow q$ in $N$. Since $f$ is a bijection, $p_{n}=f^{-1}\left(q_{n}\right)$ and $p=f^{-1}(q)$ are well defined points in $M$. To check continuity of $f^{-1}$ we must show that $p_{n} \rightarrow p$.

If ( $p_{n}$ ) refuses to converge to $p$ then there is a subsequence $\left(p_{n_{k}}\right)$ and a $\delta>0$ such that for all $k$ we have $d\left(p_{n_{k}}, p\right) \geq \delta$. Compactness of $M$ gives a sub-subsequence $\left(p_{n_{k(\ell)}}\right)$ that converges to a point $p^{*} \in M$ as $\ell \rightarrow \infty$.

Necessarily, $d\left(p, p^{*}\right) \geq \delta$, which implies that $p \neq p^{*}$. Since $f$ is continuous we have

$$
f\left(p_{n_{k(\ell)}}\right) \rightarrow f\left(p^{*}\right)
$$

as $\ell \rightarrow \infty$. The limit of a convergent sequence is unchanged by passing to a subsequence, and so $f\left(p_{n_{k(\ell)}}\right)=q_{n_{k(\ell)}} \rightarrow q$ as $\ell \rightarrow \infty$. Thus, $f\left(p^{*}\right)=q=f(p)$, contrary to $f$ being a bijection. It follows that $p_{n} \rightarrow p$ and therefore that $f^{-1}$ is continuous.

If $M$ is not compact then Theorem 40 becomes false. For example, the function $f:[0,2 \pi) \rightarrow \mathbb{R}^{2}$ defined by $f(x)=(\cos x, \sin x)$ is a continuous bijection onto the unit circle in the plane, but it is not a homeomorphism. This useful example was discussed on page 65 . Not only does this $f$ fail to be a homeomorphism, but there is no homeomorphism at all from $[0,2 \pi)$ to $S^{1}$. The circle is compact while $[0,2 \pi)$ is not. Therefore they are not homeomorphic. See also Exercises 49 and 50.

## Embedding a Compact

Not only is a compact space $M$ closed in itself, as is every metric space, but it is also a closed subset of each metric space in which it is embedded. More precisely we say that $h: M \rightarrow N$ embeds $M$ into $N$ if $h$ is a homeomorphism from $M$ onto $h M$. (The metric on $h M$ is the one it inherits from $N$.) Topologically $M$ and $h M$ are equivalent. A property of $M$ that holds also for every embedded copy of $M$ is an absolute or intrinsic property of $M$.

41 Theorem A compact is absolutely closed and absolutely bounded.
Proof Obvious from Theorems 26 and 36.

For example, no matter how the circle is embedded in $\mathbb{R}^{3}$, its image is always closed and bounded. See also Exercises 48 and 120.

## Uniform Continuity and Compactness

In Chapter 1 we defined the concept of uniform continuity for real-valued functions of a real variable. The definition in metric spaces is analogous. A function $f: M \rightarrow N$ is uniformly continuous if for each $\epsilon>0$ there exists a $\delta>0$ such that

$$
p, q \in M \text { and } d_{M}(p, q)<\delta \Rightarrow d_{N}(f p, f q)<\epsilon
$$

42 Theorem Every continuous function defined on a compact is uniformly continuous.

Proof Suppose not, and $f: M \rightarrow N$ is continuous, $M$ is compact, but $f$ fails to be uniformly continuous. Then there is some $\epsilon>0$ such that no matter how small
$\delta$ is, there exist points $p, q \in M$ with $d(p, q)<\delta$ but $d(f p, f q) \geq \epsilon$. Take $\delta=1 / n$ and let $\left(p_{n}\right)$ and $\left(q_{n}\right)$ be sequences of points in $M$ such that $d\left(p_{n}, q_{n}\right)<1 / n$ while $d\left(f\left(p_{n}\right), f\left(q_{n}\right)\right) \geq \epsilon$. Compactness of $M$ implies that there is a subsequence $p_{n_{k}}$ which converges to some $p \in M$ as $k \rightarrow \infty$. Since $d\left(p_{n}, q_{n}\right)<1 / n \rightarrow 0$ as $n \rightarrow \infty,\left(q_{n_{k}}\right)$ converges to the same limit as does $\left(p_{n_{k}}\right)$ as $k \rightarrow \infty$; namely $q_{n_{k}} \rightarrow p$. Continuity at $p$ implies that $f\left(p_{n_{k}}\right) \rightarrow f p$ and $f\left(q_{n_{k}}\right) \rightarrow f p$. If $k$ is large then

$$
d\left(f\left(p_{n_{k}}\right), f\left(q_{n_{k}}\right)\right) \leq d\left(f\left(p_{n_{k}}\right), f p\right)+d\left(f p, f\left(q_{n_{k}}\right)\right)<\epsilon,
$$

contrary to the supposition that $d\left(f\left(p_{n}\right), f\left(q_{n}\right)\right) \geq \epsilon$ for all $n$.

Theorem 42 gives a second proof that continuity implies uniform continuity on an interval $[a, b]$. For $[a, b]$ is compact.

## 5 Connectedness

As another application of these ideas, we consider the general notion of connectedness. Let $A$ be a subset of a metric space $M$. If $A$ is neither the empty set nor $M$ then $A$ is a proper subset of $M$. Recall that if $A$ is both closed and open in $M$ it is said to be clopen. The complement of a clopen set is clopen. The complement of a proper subset is proper.

If $M$ has a proper clopen subset $A$ then $M$ is disconnected. For there is a separation of $M$ into proper, disjoint clopen subsets,

$$
M=A \sqcup A^{c} .
$$

(The notation $\sqcup$ indicates disjoint union.) $M$ is connected if it is not disconnected, i.e., it contains no proper clopen subset. Connectedness of $M$ does not mean that $M$ is connected to something, but rather that $M$ is one unbroken thing. See Figure 40.


Figure $40 M$ and $N$ illustrate the difference between being connected and being disconnected.

43 Theorem If $M$ is connected, $f: M \rightarrow N$ is continuous, and $f$ is onto then $N$ is connected. The continuous image of a connected is connected.

Proof Simple! If $A$ is a clopen proper subset of $N$ then, according to the open and closed set conditions for continuity, $f^{\text {pre }}(A)$ is a clopen subset of $M$. Since $f$ is onto and $A \neq \emptyset$, we have $f^{\text {pre }}(A) \neq \emptyset$. Similarly, $f^{\text {pre }}\left(A^{c}\right) \neq \emptyset$. Therefore $f^{\text {pre }}(A)$ is a proper clopen subset of $M$, contrary to $M$ being connected. It follows that $A$ cannot exist and that $N$ is connected.

44 Corollary If $M$ is connected and $M$ is homeomorphic to $N$ then $N$ is connected. Connectedness is a topological property.

Proof $N$ is the continuous image of $M$, so Theorem 43 implies it is connected.
45 Corollary (Generalized Intermediate Value Theorem) Every continuous real-valued function defined on a connected domain has the intermediate value property.

Proof Assume that $f: M \rightarrow \mathbb{R}$ is continuous and $M$ is connected. If $f$ assumes values $\alpha<\beta$ in $\mathbb{R}$ and if it fails to assume some value $\gamma$ with $\alpha<\gamma<\beta$, then

$$
M=\{x \in M: f(x)<\gamma\} \sqcup\{x \in M: f(x)>\gamma\}
$$

is a separation of $M$, contrary to connectedness.
46 Theorem $\mathbb{R}$ is connected.

Proof If $U \subset \mathbb{R}$ is nonempty and clopen we claim that $U=\mathbb{R}$. Choose some $p \in U$ and consider the set

$$
X=\{x \in U: \text { the open interval }(p, x) \text { is contained in } U\}
$$

$X$ is nonempty since $U$ is open. Let $s$ be the supremum of $X$. If $s$ is finite (i.e., $X$ is bounded above) then $s=$ l. u. b. $X$ and $s$ is a limit of $X$. Since $X \subset U$ and $U$ is closed we have $s \in U$. Since $U$ is open there is an interval $(s-r, s+r) \subset U$, which gives an interval $(p, s+r) \subset U$, contrary to $s$ being an upper bound for $X$. Hence $s=\infty$ and $U \supset(p, \infty)$. The same reasoning gives $U \supset(-\infty, p)$, so $U=\mathbb{R}$ as claimed. Thus there are no proper clopen subsets of $\mathbb{R}$ and $\mathbb{R}$ is connected.

47 Corollary (Intermediate Value Theorem for $\mathbb{R}$ ) Every continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ has the intermediate value property.

Proof Immediate from the Generalized Intermediate Value Theorem and connectedness of $\mathbb{R}$.

48 Corollary The following metric spaces are connected: The intervals $(a, b),[a, b]$, the circle, and all capital letters of the Roman alphabet.

Proof The interval $(a, b)$ is homeomorphic to $\mathbb{R}$, while $[a, b]$ is the continuous image of $\mathbb{R}$ under the map whose graph is shown in Figure 41. The circle is the continuous image of $\mathbb{R}$ under the map $t \mapsto(\cos t, \sin t)$. Connectedness of the letters $\mathrm{A}, \ldots, \mathrm{Z}$ is equally clear.


Figure 41 The function $f$ surjects $\mathbb{R}$ continuously to $[a, b]$.
Connectedness properties give a good way to distinguish nonhomeomorphic sets. Example The union of two disjoint closed intervals is not homeomorphic to a single interval. One set is disconnected and the other is connected.

Example The closed interval $[a, b]$ is not homeomorphic to the circle $S^{1}$. For removal of a point $x \in(a, b)$ disconnects $[a, b]$ while the circle remains connected upon removal of any point. More precisely, suppose that $h:[a, b] \rightarrow S^{1}$ is a homeomorphism. Choose a point $x \in(a, b)$, and consider $X=[a, b] \backslash\{x\}$. The restriction of $h$ to $X$ is a homeomorphism from $X$ onto $Y$, where $Y$ is the circle with the point $h x$ removed. But $X$ is disconnected while $Y$ is connected. Hence $h$ cannot exist and the segment is not homeomorphic to the circle.

Example The circle is not homeomorphic to the figure eight. Removing any two points of the circle disconnects it, but this is not true of the figure eight. Or, removing
the crossing point disconnects the figure eight but removing any point of the circle leaves it connected.

Example The circle is not homeomorphic to the disc. For removing two points disconnects the circle but does not disconnect the disc.

As you can see, it is useful to be able to recognize disconnected subsets $S$ of a metric space $M$. By definition, $S$ is a disconnected subset of $M$ if it is disconnected when considered in its own right as a metric space with the metric it inherits from $M$; i.e., it has a separation $S=A \sqcup B$ such that $A$ and $B$ are proper clopen subsets of $S$. The sets $A, B$ are separated in $S$ but they need not be separated in $M$. Their closures in $M$ may intersect.

Example The punctured interval $X=[a, b] \backslash\{c\}$ is disconnected if $a<c<b$. For $X=[a, c) \sqcup(c, b]$ is a separation of $X$. The closures of the two sets with respect to the metric space $X$ do not intersect, even though their closures with respect to $\mathbb{R}$ do intersect. Pay attention to this phenomenon which is related to the Inheritance Principle.

Example Any subset $Y$ of the punctured interval is disconnected if it meets both $[a, c)$ and $(c, b]$. For $Y=([a, c) \cap Y) \sqcup((c, b] \cap Y)$ is a separation of $Y$.

49 Theorem The closure of a connected set is connected. More generally, if $S \subset M$ is connected and $S \subset T \subset \bar{S}$ then $T$ is connected.

Proof It is equivalent to show that if $T$ is disconnected then $S$ is disconnected. Disconnectedness of $T$ implies that

$$
T=A \sqcup B
$$

where $A, B$ are clopen and proper in $T$. It is natural to expect that

$$
S=K \sqcup L
$$

is a separation of $S$ where $K=A \cap S$ and $L=B \cap S$. The sets $K$ and $L$ are disjoint, their union is $S$, and by the Inheritance Principle they are clopen. But are they proper?

If $K=\emptyset$ then $A \subset S^{c}$. Since $A$ is proper there exists $p \in A$. Since $A$ is open in $T$, there exists a neighborhood $M_{r} p$ such that

$$
T \cap M_{r} p \subset A \subset S^{c}
$$

The neighborhood $M_{r} p$ contains no points of $S$, which is contrary to $p$ belonging to $\bar{S}$. Thus, $K \neq \emptyset$. Similarly, $L=B \cap S \neq \emptyset$, so $S=K \sqcup L$ is a separation of $S$, proving that $S$ is disconnected.

Example The outward spiral expressed in polar coordinates as

$$
S=\{(r, \theta):(1-r) \theta=1 \text { and } \theta \geq \pi / 2\}
$$

has $\bar{S}=S \cup S^{1}$, where $S^{1}$ is the unit circle. Since $S$ is connected, so is $\bar{S}$. (Recall that $\bar{S}$ is the closure of $S$.) See Figure 27.

50 Theorem The union of connected sets sharing a common point $p$ is connected.

Proof Let $S=\mathbf{U} S_{\alpha}$, where each $S_{\alpha}$ is connected and $p \in \bigcap S_{\alpha}$. If $S$ is disconnected then it has a separation $S=A \sqcup A^{c}$ where $A, A^{c}$ are proper and clopen. One of them contains $p$; say it is $A$. Then $A \cap S_{\alpha}$ is a nonempty clopen subset of $S_{\alpha}$. Since $S_{\alpha}$ is connected, $A \cap S_{\alpha}=S_{\alpha}$ for each $\alpha$, and $A=S$. This implies that $A^{c}=\emptyset$, a contradiction. Therefore $S$ is connected.

Example The 2-sphere $S^{2}$ is connected. For $S^{2}$ is the union of great circles, each passing through the poles.

Example Every convex set $C$ in $\mathbb{R}^{m}$ (or in any metric space with a compatible linear structure) is connected. If we choose a point $p \in C$ then each $q \in C$ lies on a line segment $[p, q] \subset C$. Thus, $C$ is the union of connected sets sharing the common point $p$. It is connected.

Definition A path joining $p$ to $q$ in a metric space $M$ is a continuous function $f:[a, b] \rightarrow M$ such that $f a=p$ and $f b=q$. If each pair of points in $M$ can be joined by a path in $M$ then $M$ is path-connected. See Figure 42.

## 51 Theorem Path-connected implies connected.

Proof Assume that $M$ is path-connected but not connected. Then $M=A \sqcup A^{c}$ for some proper clopen $A \subset M$. Choose $p \in A$ and $q \in A^{c}$. There is a path $f:[a, b] \rightarrow M$ from $p$ to $q$. The separation $f^{\text {pre }}(A) \sqcup f^{\text {pre }}\left(A^{c}\right)$ contradicts connectedness of $[a, b]$. Therefore $M$ is connected.

Example All connected subsets of $\mathbb{R}$ are path-connected. See Exercise 67.


Figure 42 A path $f$ in $M$ that joins $p$ to $q$

Example Every open connected subset of $\mathbb{R}^{m}$ is path-connected. See Exercises 61 and 66.

Example The topologist's sine curve is a compact connected set that is not path-connected. It is $M=G \cup Y$ where

$$
\begin{aligned}
G & =\left\{(x, y) \in \mathbb{R}^{2}: y=\sin 1 / x \text { and } 0<x \leq 1 / \pi\right\} \\
Y & =\left\{(0, y) \in \mathbb{R}^{2}:-1 \leq y \leq 1\right\}
\end{aligned}
$$

See Figure 43. The metric on $M$ is just Euclidean distance. Is $M$ connected? Yes!


Figure 43 The topologist's sine curve $M$ is a connected set. It includes the vertical segment $Y$ at $x=0$.

The graph $G$ is connected and $M=\bar{G}$. By Theorem $49 M$ is connected.

## 6 Other Metric Space Concepts

Here are a few standard metric space topics related to what appears above. If $S \subset M$ then its closure is the smallest closed subset of $M$ that contains $S$, its interior is the largest open subset of $M$ contained in $S$, and its boundary is the difference between its closure and its interior. Their notations are

$$
\bar{S}=\operatorname{cl} S=\text { closure of } S \quad \operatorname{int} S=\text { interior of } S \quad \partial S=\text { boundary of } S
$$

To avoid inheritance ambiguity it would be better (but too cumbersome) to write $\mathrm{cl}_{M} S$, $\operatorname{int}_{M} S$, and $\partial_{M} S$ to indicate the ambient space $M$. In Exercise 95 you are asked to check various simple facts about them, such as $\bar{S}=\lim S=$ the intersection of all closed sets that contain $S$.

## Clustering and Condensing

Two concepts similar to limits are clustering and condensing. The set $S$ "clusters" at $p$ (and $p$ is a cluster point ${ }^{\dagger}$ of $S$ ) if each $M_{r} p$ contains infinitely many points of $S$. The set $S$ condenses at $p$ (and $p$ is a condensation point of $S$ ) if each $M_{r} p$ contains uncountably many points of $S$. Thus, $S$ limits at $p$, clusters at $p$, or condenses at $p$ according to whether each $M_{r} p$ contains some, infinitely many, or uncountably many points of $S$. See Figure 44.


Figure 44 Limiting, clustering, and condensing behavior

[^6]52 Theorem The following are equivalent conditions to $S$ clustering at $p$.
(i) There is a sequence of distinct points in $S$ that converges to $p$.
(ii) Each neighborhood of $p$ contains infinitely many points of $S$.
(iii) Each neighborhood of $p$ contains at least two points of $S$.
(iv) Each neighborhood of $p$ contains at least one point of $S$ other than $p$.

Proof Clearly (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv), and (ii) is the definition of clustering. It remains to check (iv) $\Rightarrow$ (i).

Assume (iv) is true: Each neighborhood of $p$ contains a point of $S$ other than p. In $M_{1} p$ choose a point $p_{1} \in(S \backslash\{p\})$. Set $r_{2}=\min \left(1 / 2, d\left(p_{1}, p\right)\right)$, and in the smaller neighborhood $M_{r_{2}} p$, choose $p_{2} \in(S \backslash\{p\})$. Proceed inductively: Set $r_{n}=\min \left(1 / n, d\left(p_{n-1}, p\right)\right)$ and in $M_{r_{n}} p$, choose $p_{n} \in(S \backslash\{p\})$. Since $r_{n} \rightarrow 0$ the sequence $\left(p_{n}\right)$ converges to $p$. The points $p_{n}$ are distinct since they have different distances to $p$,

$$
d\left(p_{1}, p\right) \geq r_{2}>d\left(p_{2}, p\right) \geq r_{3}>d\left(p_{3}, p\right) \geq \ldots
$$

Thus (iv) $\Rightarrow$ (i) and the four conditions are equivalent.

Condition (iv) is the form of the definition of clustering most frequently used, although it is the hardest to grasp. It is customary to denote by $S^{\prime}$ the set of cluster points of $S$.

53 Proposition $S \cup S^{\prime}=\bar{S}$.
Proof A cluster point is a type of limit of $S$, so $S^{\prime} \subset \lim S=\bar{S}$ and

$$
S \cup S^{\prime} \subset \bar{S}
$$

On the other hand, if $p \in \bar{S}$ then either $p \in S$ or else $p \notin S$ and each neighborhood of $p$ contains points of $S$ other than $p$. This implies that $p \in S \cup S^{\prime}$, so $\bar{S} \subset S \cup S^{\prime}$, and the two sets are equal.

54 Corollary $S$ is closed if and only if $S^{\prime} \subset S$.
Proof $S$ is closed if and only if $S=\bar{S}$. Since $\bar{S}=S \cup S^{\prime}$, equivalent to $S^{\prime} \subset S$ is $\bar{S}=S$.

55 Corollary The least upper bound and greatest lower bound of a nonempty bounded set $S \subset \mathbb{R}$ belong to the closure of $S$. Thus, if $S$ is closed then they belong to $S$.

Proof If $b=$ l. u. b. $S$ then each interval $(b-r, b]$ contains points of $S$. The same is true for intervals $[a, a+r)$ where $a=$ g.l.b. $S$

## Perfect Metric Spaces

A metric space $M$ is perfect if $M^{\prime}=M$, i.e., each $p \in M$ is a cluster point of $M$. Recall that $M$ clusters at $p$ if each $M_{r} p$ is an infinite set. For example $[a, b]$ is perfect and $\mathbb{Q}$ is perfect. $\mathbb{N}$ is not perfect since none of its points are cluster points.

56 Theorem Every nonempty, perfect, complete metric space is uncountable.
Proof Suppose not: Assume $M$ is nonempty, perfect, complete, and countable. Since $M$ consists of cluster points it must be denumerable and not finite. Say

$$
M=\left\{x_{1}, x_{2}, \ldots\right\}
$$

is a list of all the elements of $M$. We will derive a contradiction by finding a point of $M$ not in the list. Define

$$
\widehat{M}_{r} p=\{q \in M: d(p, q) \leq r\}
$$

It is the closed neighborhood of radius $r$ at $p$. Choose any $y_{1} \in M$ with $y_{1} \neq x_{1}$ and choose $r_{1}>0$ so that $Y_{1}=\widehat{M}_{r_{1}}\left(y_{1}\right)$ "excludes" $x_{1}$ in the sense that $x_{1} \notin Y_{1}$. We can take $r_{1}$ as small as we want, say $r_{1}<1$.

Since $M$ clusters at $y_{1}$ we can choose $y_{2} \in M_{r_{1}}\left(y_{1}\right)$ with $y_{2} \neq x_{2}$ and choose $r_{2}>0$ so that $Y_{2}=\widehat{M}_{r_{2}}\left(y_{2}\right)$ excludes $x_{2}$. Taking $r_{2}$ small ensures $Y_{2} \subset Y_{1}$. (Here we are using openness of $M_{r_{1}}\left(y_{1}\right)$.) Also we take $r_{2}<1 / 2$. Since $Y_{2} \subset Y_{1}$, it excludes $x_{1}$ as well as $x_{2}$. See Figure 45 .

Nothing stops us from continuing inductively, and we get a nested sequence of closed neighborhoods $Y_{1} \supset Y_{2} \supset Y_{3} \ldots$ such that $Y_{n}$ excludes $x_{1}, \ldots, x_{n}$, and has radius $r_{n} \leq 1 / n$. Thus the center points $y_{n}$ form a Cauchy sequence. Completeness of $M$ implies that

$$
\lim _{n \rightarrow \infty} y_{n}=y \in M
$$

exists. Since the sets $Y_{n}$ are closed and nested, $y \in Y_{n}$ for each $n$. Does $y$ equal $x_{1}$ ? No, for $Y_{1}$ excludes $x_{1}$. Does it equal $x_{2}$ ? No, for $Y_{2}$ excludes $x_{2}$. In fact, for each $n$ we have $y \neq x_{n}$. The point $y$ is nowhere in the supposedly complete list of elements of $M$, a contradiction. Hence $M$ is uncountable.

57 Corollary $\mathbb{R}$ and $[a, b]$ are uncountable.
Proof $\mathbb{R}$ is complete and perfect, while $[a, b]$ is compact, therefore complete, and perfect. Neither is empty.


Figure 45 The exclusion of successively more points of the sequence $\left(x_{n}\right)$ that supposedly lists all the elements of $M$

58 Corollary Every nonempty perfect complete metric space is everywhere uncountable in the sense that each r-neighborhood is uncountable.

Proof The $r / 2$-neighborhood $M_{r / 2}(p)$ is perfect: It clusters at each of its points. The closure of a perfect set is perfect. Thus, $\overline{M_{r / 2}(p)}$ is perfect. Being a closed subset of a complete metric space, it is complete. According to Theorem 56, $\overline{M_{r / 2}(p)}$ is uncountable. Since $\overline{M_{r / 2}(p)} \subset M_{r} p, M_{r} p$ is uncountable.

## Continuity of Arithmetic in $\mathbb{R}$

Addition is a mapping Sum : $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ that assigns to $(x, y)$ the real number $x+y$. Subtraction and multiplication are also such mappings. Division is a mapping $\mathbb{R} \times(\mathbb{R} \backslash\{0\}) \rightarrow \mathbb{R}$ that assigns to $(x, y)$ the number $x / y$.

59 Theorem The arithmetic operations of $\mathbb{R}$ are continuous.

60 Lemma For each real number $c$ the function $\operatorname{Mult}_{c}: \mathbb{R} \rightarrow \mathbb{R}$ that sends $x$ to $c x$ is continuous.

Proof If $c=0$ the function is constantly equal to 0 and is therefore continuous. If $c \neq 0$ and $\epsilon>0$ is given, choose $\delta=\epsilon /|c|$. If $|x-y|<\delta$ then

$$
\left|\operatorname{Mult}_{c}(x)-\operatorname{Mult}_{c}(y)\right|=|c||x-y|<|c| \delta=\epsilon
$$

which shows that Mult ${ }_{c}$ is continuous.
Proof of Theorem 59 We use the preservation of sequential convergence criterion for continuity. It's simplest. Let $\left(x_{n}, y_{n}\right) \rightarrow(x, y)$ as $n \rightarrow \infty$.

By the triangle inequality we have

$$
\left|\operatorname{Sum}\left(x_{n}, y_{n}\right)-\operatorname{Sum}(x, y)\right| \leq\left|x_{n}-x\right|+\left|y_{n}-y\right|=d_{\text {sum }}\left(\left(x_{n}, y_{n}\right),(x, y)\right)
$$

By Corollary $21 d_{\text {sum }}$ is continuous, so $d_{\text {sum }}\left(\left(x_{n}, y_{n}\right),(x, y)\right) \rightarrow 0$ as $n \rightarrow \infty$, which completes the proof that Sum is continuous. (By Theorem 17 it does not matter which metric we use on $\mathbb{R} \times \mathbb{R}$.)

Subtraction is the composition of continuous functions

$$
\operatorname{Sub}(x, y)=\operatorname{Sum} \circ\left(\mathrm{id} \times \operatorname{Mult}_{-1}\right)(x, y)
$$

and is therefore continuous. (Proposition 3 implies id is continuous, Lemma 60 implies Mult -1 is continuous, and Corollary 18 implies id $\times \mathrm{Mult}_{-1}$ is continuous.)

Multiplication is continuous since

$$
\begin{aligned}
\left|\operatorname{Mult}\left(x_{n}, y_{n}\right)-\operatorname{Mult}(x, y)\right| & =\left|x_{n} y_{n}-x y\right| \\
& \leq\left|x_{n}-x\right|\left|y_{n}\right|+|x|\left|y_{n}-y\right| \\
& \leq B\left(\left|x-x_{n}\right|+\left|y-y_{n}\right|\right) \\
& =\operatorname{Mult}_{B}\left(d_{\text {sum }}\left(\left(x_{n}, y_{n}\right),(x, y)\right)\right) \rightarrow 0
\end{aligned}
$$

as $n \rightarrow \infty$, where we use the fact that convergent sequences are bounded to write $\left|y_{n}\right|+|x| \leq B$ for all $n$.

Reciprocation is the function Rec $: \mathbb{R} \backslash\{0\} \rightarrow \mathbb{R} \backslash\{0\}$ that sends $x$ to $1 / x$. If $x_{n} \rightarrow x \neq 0$ then there is a constant $b>0$ such that for all large $n$ we have $\left|1 / x_{n}\right| \leq b$ and $|1 / x| \leq b$. Since

$$
\left|\operatorname{Rec}\left(x_{n}\right)-\operatorname{Rec}(x)\right|=\left|\frac{1}{x_{n}}-\frac{1}{x}\right|=\frac{\left|x_{n}-x\right|}{\left|x x_{n}\right|} \leq \operatorname{Mult}_{b^{2}}\left(\left|x_{n}-x\right|\right) \rightarrow 0
$$

as $n \rightarrow \infty$ we see that Rec is continuous.
Division is continuous on $\mathbb{R} \times(\mathbb{R} \backslash\{0\})$ since it is the composite of continuous mappings Mult $\circ(\mathrm{id} \times \operatorname{Rec}):(x, y) \mapsto(x, 1 / y) \mapsto x \cdot 1 / y$.

The absolute value is a mapping Abs : $\mathbb{R} \rightarrow \mathbb{R}$ that sends $x$ to $|x|$. It is continuous since it is $d(x, 0)$ and the distance function is continuous. The maximum and minimum are functions $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ given by the formulas

$$
\max (x, y)=\frac{x+y}{2}+\frac{|x-y|}{2} \quad \min (x, y)=\frac{x+y}{2}-\frac{|x-y|}{2}
$$

so they are also continuous.
61 Corollary The sums, differences, products, and quotients, absolute values, maxima, and minima of real-valued continuous functions are continuous. (The denominator functions should not equal zero.)

Proof Take, for example, the sum $f+g$ where $f, g: M \rightarrow \mathbb{R}$ are continuous. It is the composite of continuous functions

$$
\left.\begin{array}{rl}
M & \xrightarrow{f \times g} \mathbb{R} \times \mathbb{R} \quad \xrightarrow{\text { Sum }} \mathbb{R} \\
x & \mapsto(f x, g x)
\end{array}\right) \quad \operatorname{Sum}(f x, g x),
$$

and is therefore continuous. The same applies to the other operations.
62 Corollary Polynomials are continuous functions.
Proof Proposition 3 states that constant functions and the identity function are continuous. Thus Corollary 61 and induction imply that the polynomial $a_{0}+a_{1} x+\cdots+$ $a_{n} x^{n}$ is continuous.

The same reasoning shows that polynomials of $m$ variables are continuous functions $\mathbb{R}^{m} \rightarrow \mathbb{R}$.

## Boundedness

A subset $S$ of a metric space $M$ is bounded if for some $p \in M$ and some $r>0$,

$$
S \subset M_{r} p
$$

A set which is not bounded is unbounded. For example, the elliptical region $4 x^{2}+$ $y^{2}<4$ is a bounded subset of $\mathbb{R}^{2}$, while the hyperbola $x y=1$ is unbounded. It is easy to see that if $S$ is bounded then for each $q \in M$ there is an $s$ such that $M_{s} q$ contains $S$.

Distinguish the word "bounded" from the word "finite." The first refers to physical size, the second to the number of elements. The concepts are totally different.

Also, boundedness has little connection to the existence of a boundary - a clopen subset of a metric space has empty boundary, but some clopen sets are bounded, others not.

Exercise 39 asks you to show that every convergent sequence is bounded, and to decide whether it is also true that every Cauchy sequence is bounded, even when the metric space is not complete.

Boundedness is not a topological property. For example, $(-1,1)$ and $\mathbb{R}$ are homeomorphic although $(-1,1)$ is bounded and $\mathbb{R}$ is unbounded. The same example shows that completeness is not a topological property.

A function from $M$ to another metric space $N$ is a bounded function if its range is a bounded subset of $N$. That is, there exist $q \in N$ and $r>0$ such that

$$
f M \subset N_{r} q
$$

Note that a function can be bounded even though its graph is not. For example, $x \mapsto \sin x$ is a bounded function $\mathbb{R} \rightarrow \mathbb{R}$ although its graph, $\left\{(x, y) \in \mathbb{R}^{2}: y=\sin x\right\}$, is an unbounded subset of $\mathbb{R}^{2}$.

## 7 Coverings

For the sake of simplicity we have postponed discussing compactness in terms of open coverings until this point. Typically, students find coverings a challenging concept. It is central, however, to much of analysis - for example, measure theory.

Definition A collection $\mathcal{U}$ of subsets of $M$ covers $A \subset M$ if $A$ is contained in the union of the sets belonging to $\mathcal{U}$. The collection $\mathcal{U}$ is a covering of $A$. If $\mathcal{U}$ and $\mathcal{V}$ both cover $A$ and if $\mathcal{V} \subset \mathcal{U}$ in the sense that each set $V \in \mathcal{V}$ belongs also to $\mathcal{U}$ then we say that $\mathcal{U}$ reduces to $\mathcal{V}$, and that $\mathcal{V}$ is a subcovering of $A$.

Definition If all the sets in a covering $\mathcal{U}$ of $A$ are open then $\mathcal{U}$ is an open covering of $A$. If every open covering of $A$ reduces to a finite subcovering of $A$ then we say that $A$ is covering compact ${ }^{\dagger}$.

The idea is that if $A$ is covering compact and $\mathcal{U}$ is an open covering of $A$ then just a finite number of the open sets are actually doing the work of covering $A$. The rest are redundant.

[^7]A covering $\mathcal{U}$ of $A$ is also called a cover of $A$. The members of $\mathcal{U}$ are not called covers. Instead, you could call them scraps or patches. Imagine the covering as a patchwork quilt that covers a bed, the quilt being sewn together from overlapping scraps of cloth. See Figure 46.


Figure 46 A covering of $A$ by eight scraps. The set $A$ is cross-hatched. The scraps are two discs, two rectangles, two ellipses, a pentagon, and a triangle. Each point of $A$ belongs to at least one scrap.

The mere existence of a finite open covering of $A$ is trivial and utterly worthless. Every set $A$ has such a covering, namely the single open set $M$. Rather, for $A$ to be covering compact, each and every open covering of $A$ must reduce to a finite subcovering of $A$. Deciding directly whether this is so is daunting. How could you hope to verify the finite reducibility of all open coverings of $A$ ? There are so many of them. For this reason we concentrated on sequential compactness; it is relatively easy to check by inspection whether every sequence in a set has a convergent subsequence.

To check that a set is not covering compact it suffices to find an open covering which fails to reduce to a finite subcovering. Occasionally this is simple. For example,
the set $(0,1]$ is not covering compact in $\mathbb{R}$ because its covering

$$
\mathcal{U}=\{(1 / n, 2): n \in \mathbb{N}\}
$$

fails to reduce to a finite subcovering.
63 Theorem For a subset $A$ of a metric space $M$ the following are equivalent:
(a) $A$ is covering compact.
(b) $A$ is sequentially compact.

Proof that (a) implies (b) We assume $A$ is covering compact and prove it is sequentially compact. Suppose not. Then there is a sequence $\left(p_{n}\right)$ in $A$, no subsequence of which converges in $A$. Each point $a \in A$ therefore has some neighborhood $M_{r} a$ such that $p_{n} \in M_{r} a$ only finitely often. (The radius $r$ may depend on the point $a$.) The collection $\left\{M_{r} a: a \in A\right\}$ is an open covering of $A$ and by covering compactness it reduces to a finite subcovering

$$
\left\{M_{r_{1}}\left(a_{1}\right), M_{r_{2}}\left(a_{2}\right), \ldots, M_{r_{k}}\left(a_{k}\right)\right\}
$$

of $A$. Since $p_{n}$ appears in each of these finitely many neighborhoods $M_{r_{i}}\left(a_{i}\right)$ only finitely often, it follows from the pigeonhole principle that $\left(p_{n}\right)$ has only finitely many terms, a contradiction. Thus $\left(p_{n}\right)$ cannot exist, and $A$ is sequentially compact.

The following presentation of the proof that (b) implies (a) appears in Royden's book, Real Analysis. A Lebesgue number for a covering $\mathcal{U}$ of $A$ is a positive real number $\lambda$ such that for each $a \in A$ there is some $U \in \mathcal{U}$ with $M_{\lambda} a \subset U$. Of course, the choice of this $U$ depends on $a$. It is crucial, however, that the Lebesgue number $\lambda$ is independent of $a \in A$.

The idea of a Lebesgue number is that we know each point $a \in A$ is contained in some $U \in \mathcal{U}$, and if $\lambda$ is extremely small then $M_{\lambda} a$ is just a slightly swollen point so the same should be true for it too. No matter where in $A$ the neighborhood $M_{\lambda} a$ is placed, it should lie wholly in some member of the covering. See Figure 47.

If $A$ is noncompact then it may have open coverings with no positive Lebesgue number. For example, let $A=(0,1) \subset \mathbb{R}=M$. The singleton collection $\{A\}$ is an open covering of $A$, but there is no $\lambda>0$ such that for every $a \in A$ we have $(a-\lambda, a+\lambda) \subset A$. See Exercise 86 .

64 Lebesgue Number Lemma Every open covering of a sequentially compact set has a Lebesgue number $\lambda>0$.


Figure 47 Small neighborhoods are like swollen points. $\mathcal{U}$ has a positive Lebesgue number $\lambda$. The $\lambda$-neighborhood of each point in the cross-hatched set $A$ is wholly contained in at least one member of the covering.

Proof Suppose not: $\mathcal{U}$ is an open covering of a sequentially compact set $A$, and yet for each $\lambda>0$ there exists an $a \in A$ such that no $U \in \mathcal{U}$ contains $M_{\lambda} a$. Take $\lambda=1 / n$ and let $a_{n} \in A$ be a point such that no $U \in \mathcal{U}$ contains $M_{1 / n}\left(a_{n}\right)$. By sequential compactness, there is a subsequence $\left(a_{n_{k}}\right)$ converging to some point $p \in A$. Since $\mathcal{U}$ is an open covering of $A$, there exist $r>0$ and $U \in \mathcal{U}$ with $M_{r} p \subset U$. If $k$ is large then $d\left(a_{n_{k}}, p\right)<r / 2$ and $1 / n_{k}<r / 2$, which implies by the triangle inequality that

$$
M_{1 / n_{k}}\left(a_{n_{k}}\right) \subset M_{r} p \subset U,
$$

contrary to the supposition that no $U \in \mathcal{U}$ contains $M_{1 / n}\left(a_{n}\right)$. We conclude that, after all, $\mathcal{U}$ does have a Lebesgue number $\lambda>0$. See Figure 48 .

Proof that (b) implies (a) in Theorem 63 Let $\mathcal{U}$ be an open covering of the sequentially compact set $A$. We want to reduce $\mathcal{U}$ to a finite subcovering. By the Lebesgue Number Lemma, $\mathcal{U}$ has a Lebesgue number $\lambda>0$. Choose any $a_{1} \in A$ and some $U_{1} \in \mathcal{U}$ such that

$$
M_{\lambda}\left(a_{1}\right) \subset U_{1}
$$



Figure 48 The neighborhood $M_{r} p$ engulfs the smaller neighborhood

$$
M_{1 / n_{k}}\left(a_{n_{k}}\right) .
$$

If $U_{1} \supset A$ then $\mathcal{U}$ reduces to the finite subcovering $\left\{U_{1}\right\}$ consisting of a single set, and the implication $(\mathrm{b}) \Rightarrow(\mathrm{a})$ is proved. On the other hand, as is more likely, if $U_{1}$ does not contain $A$ then we choose a point $a_{2} \in A \backslash U_{1}$ and $U_{2} \in \mathcal{U}$ such that

$$
M_{\lambda}\left(a_{2}\right) \subset U_{2}
$$

Either $\mathcal{U}$ reduces to the finite subcovering $\left\{U_{1}, U_{2}\right\}$ (and the proof is finished) or else we can continue, eventually producing a sequence $\left(a_{n}\right)$ in $A$ and a sequence $\left(U_{n}\right)$ in $\mathcal{U}$ such that

$$
M_{\lambda}\left(a_{n}\right) \subset U_{n} \text { and } a_{n+1} \in\left(A \backslash\left(U_{1} \cup \cdots \cup U_{n}\right)\right)
$$

We will show that such sequences $\left(a_{n}\right),\left(U_{n}\right)$ lead to a contradiction. By sequential compactness, there is a subsequence $\left(a_{n_{k}}\right)$ that converges to some $p \in A$. For a large $k$ we have $d\left(a_{n_{k}}, p\right)<\lambda$ and

$$
p \in M_{\lambda}\left(a_{n_{k}}\right) \subset U_{n_{k}}
$$

See Figure 49.
All $a_{n_{\ell}}$ with $\ell>k$ lie outside $U_{n_{k}}$, which contradicts their convergence to $p$. Thus, at some finite stage the process of choosing points $a_{n}$ and sets $U_{n}$ terminates, and $\mathcal{U}$


Figure 49 The point $a_{n_{k}}$ is so near $p$ that the neighborhood $M_{\lambda}\left(a_{n_{k}}\right)$ engulfs $p$.
reduces to a finite subcovering $\left\{U_{1}, \ldots, U_{n}\right\}$ of $A$, which implies that $A$ is covering compact. See also the remark on page 421.

Upshot In light of Theorem 63, the term "compact" may now be applied equally to any set obeying (a) or (b).

## Total Boundedness

The Heine-Borel Theorem states that a subset of $\mathbb{R}^{m}$ is compact if and only if it is closed and bounded. In more general metric spaces, such as $\mathbb{Q}$, the assertion is false. But what if the metric space is complete? As remarked on page 81 it is still false.

But mathematicians do not quit easily. The Heine-Borel Theorem ought to generalize beyond $\mathbb{R}^{m}$ somehow. Here is the concept we need: A set $A \subset M$ is totally bounded if for each $\epsilon>0$ there exists a finite covering of $A$ by $\epsilon$-neighborhoods. No mention is made of a covering reducing to a subcovering. How close total boundedness is to the worthless fact that every metric space has a finite open covering!

65 Generalized Heine-Borel Theorem A subset of a complete metric space is compact if and only if it is closed and totally bounded.

Proof Let $A$ be a compact subset of $M$. Therefore it is closed. To see that it is totally bounded, let $\epsilon>0$ be given and consider the covering of $A$ by $\epsilon$-neighborhoods,

$$
\left\{M_{\epsilon} x: x \in A\right\} .
$$

Compactness of $A$ implies that this covering reduces to a finite subcovering and therefore $A$ is totally bounded.

Conversely, assume that $A$ is a closed and totally bounded subset of the complete metric space $M$. We claim that $A$ is sequentially compact. That is, every sequence $\left(a_{n}\right)$ in $A$ has a subsequence that converges in $A$. Set $\epsilon_{k}=1 / k, k=1,2, \ldots$ Since $A$ is totally bounded we can cover it by finitely many $\epsilon_{1}$-neighborhoods

$$
M_{\epsilon_{1}}\left(q_{1}\right), \ldots, M_{\epsilon_{1}}\left(q_{m}\right)
$$

By the pigeonhole principle, terms of the sequence $a_{n}$ lie in at least one of these neighborhoods infinitely often, say it is $M_{\epsilon_{1}}\left(p_{1}\right)$. Choose

$$
a_{n_{1}} \in A_{1}=A \cap M_{\epsilon_{1}}\left(p_{1}\right)
$$

Every subset of a totally bounded set is totally bounded, so we can cover $A_{1}$ by finitely many $\epsilon_{2}$-neighborhoods. For one of them, say $M_{\epsilon_{2}}\left(p_{2}\right), a_{n}$ lies in $A_{2}=A_{1} \cap M_{\epsilon_{2}}\left(p_{2}\right)$ infinitely often. Choose $a_{n_{2}} \in A_{2}$ with $n_{2}>n_{1}$.

Proceeding inductively, cover $A_{k-1}$ by finitely many $\epsilon_{k}$-neighborhoods, one of which, say $M_{\epsilon_{k}}\left(p_{k}\right)$, contains terms of the sequence $\left(a_{n}\right)$ infinitely often. Then choose $a_{n_{k}} \in A_{k}=A_{k-1} \cap M_{\epsilon_{k}}\left(p_{k}\right)$ with $n_{k}>n_{k-1}$. Then $\left(a_{n_{k}}\right)$ is a subsequence of $\left(a_{n}\right)$. It is Cauchy. For if $\epsilon>0$ is given we choose $N$ such that $2 / N<\epsilon$. If $k, \ell \geq N$ then

$$
a_{n_{k}}, a_{n_{\ell}} \in A_{N} \quad \text { and } \quad \operatorname{diam} A_{N} \leq 2 \epsilon_{N}=\frac{2}{N}<\epsilon
$$

which shows that $\left(a_{n_{k}}\right)$ is Cauchy. Completeness of $M$ implies that $\left(a_{n_{k}}\right)$ converges to some $p \in M$ and since $A$ is closed we have $p \in A$. Hence $A$ is compact.

66 Corollary $A$ metric space is compact if and only if it is complete and totally bounded.

Proof Every compact metric space $M$ is complete. This is because, given a Cauchy sequence $\left(p_{n}\right)$ in $M$, compactness implies that some subsequence converges in $M$, and if a Cauchy sequence has a convergent subsequence then the mother sequence converges too. As observed above, compactness immediately gives total boundedness.

Conversely, assume that $M$ is complete and totally bounded. Every metric space is closed in itself. By Theorem 65, $M$ is compact.

## 8 Cantor Sets

Cantor sets are fascinating examples of compact sets that are maximally disconnected. (To emphasize the disconnectedness, one sometimes refers to a Cantor set as "Cantor dust.") Here is how to construct the standard Cantor set. Start with the unit interval $[0,1]$ and remove its open middle third, $(1 / 3,2 / 3)$. Then remove the open middle third from the remaining two intervals, and so on. This gives a nested sequence $C^{0} \supset C^{1} \supset C^{2} \supset \ldots$ where $C^{0}=[0,1], C^{1}$ is the union of the two intervals $[0,1 / 3]$ and $[2 / 3,1], C^{2}$ is the union of four intervals $[0,1 / 9],[2 / 9,1 / 3],[2 / 3,7 / 9]$, and $[8 / 9,1]$, $C^{3}$ is the union of eight intervals, and so on. See Figure 50.


Figure 50 The construction of the standard middle-thirds Cantor set $C$

In general $C^{n}$ is the union of $2^{n}$ closed intervals, each of length $1 / 3^{n}$. Each $C^{n}$ is compact. The standard middle thirds Cantor set is the nested intersection

$$
C=\bigcap_{n=0}^{\infty} C^{n}
$$

We refer to $C$ as "the" Cantor set. Clearly it contains the endpoints of each of the intervals comprising $C^{n}$. Actually, it contains uncountably many more points than these endpoints! There are other Cantor sets defined by removing, say, middle fourths, pairs of middle tenths, etc. All Cantor sets turn out to be homeomorphic to the standard Cantor set. See Section 9.

A metric space $M$ is totally disconnected if each point $p \in M$ has arbitrarily small clopen neighborhoods. That is, given $\epsilon>0$ and $p \in M$, there exists a clopen set $U$ such that

$$
p \in U \subset M_{\epsilon} p
$$

For example, every discrete space is totally disconnected. So is $\mathbb{Q}$.
67 Theorem The Cantor set is a compact, nonempty, perfect, and totally disconnected metric space.

Proof The metric on $C$ is the one it inherits from $\mathbb{R}$, the usual distance $|x-y|$. Let $E$ be the set of endpoints of all the $C^{n}$-intervals,

$$
E=\{0,1,1 / 3,2 / 3,1 / 9,2 / 9,7 / 9,8 / 9, \ldots\}
$$

Clearly $E$ is denumerable and contained in $C$, so $C$ is nonempty and infinite. It is compact because it is the intersection of compacts.

To show $C$ is perfect and totally disconnected, take any $x \in C$ and any $\epsilon>0$. Fix $n$ so large that $1 / 3^{n}<\epsilon$. The point $x$ lies in one of the $2^{n}$ intervals $I$ of length $1 / 3^{n}$ that comprise $C^{n}$. Fix this $I$. The set $E \cap I$ is infinite and contained in the interval $(x-\epsilon, x+\epsilon)$. Thus $C$ clusters at $x$ and $C$ is perfect. See Figure 51.


Figure 51 The endpoints of $C$ cluster at $x$.

The interval $I$ is closed in $\mathbb{R}$ and therefore in $C^{n}$. The complement $J=C^{n} \backslash I$ consists of finitely many closed intervals and is therefore closed too. Thus, $I$ and $J$ are clopen in $C^{n}$. By the Inheritance Principle their intersections with $C$ are clopen in $C$, so $C \cap I$ is a clopen neighborhood of $x$ in $C$ which is contained in the $\epsilon$-neighborhood of $x$, completing the proof that $C$ is totally disconnected.

68 Corollary The Cantor set is uncountable.
Proof Being compact, $C$ is complete, and by Theorem 56, every complete, perfect, nonempty metric space is uncountable.

A more direct way to see that the Cantor set is uncountable involves a geometric coding scheme. Take the code $0=$ left and $2=$ right. Then

$$
C_{0}=\text { left interval }=[0,1 / 3] \quad C_{2}=\text { right interval }=[2 / 3,1]
$$

and $C^{1}=C_{0} \cup C_{2}$. Similarly, the left and right subintervals of $C_{0}$ are coded $C_{00}$ and $C_{02}$, while the left and right subintervals of $C_{2}$ are $C_{20}$ and $C_{22}$. This gives

$$
C^{2}=C_{00} \sqcup C_{02} \quad \sqcup \quad C_{20} \sqcup C_{22} .
$$

The intervals that comprise $C^{3}$ are specified by strings of length 3 . For instance, $C_{220}$ is the left subinterval of $C_{22}$. In general an interval of $C^{n}$ is coded by an address string of $n$ symbols, each a 0 or a 2 . Read it like a zip code. The first symbol gives the interval's gross location (left or right), the second symbol refines the location, the third refines it more, and so on.

Imagine now an infinite address string $\omega=\omega_{1} \omega_{2} \omega_{3} \ldots$ of zeros and twos. Corresponding to $\omega$, we form a nested sequence of intervals

$$
C_{\omega_{1}} \supset C_{\omega_{1} \omega_{2}} \supset C_{\omega_{1} \omega_{2} \omega_{3}} \supset \cdots \supset C_{\omega_{1} \ldots \omega_{n}} \supset \ldots,
$$

the intersection of which is a point $p=p(\omega) \in C$. Specifically,

$$
p(\omega)=\bigcap_{n \in \mathbb{N}} C_{\omega \mid n}
$$

where $\omega \mid n=\omega_{1} \ldots \omega_{n}$ truncates $\omega$ to an address of length $n$. See Theorem 34 .
As we have observed, each infinite address string defines a point in the Cantor set. Conversely, each point $p \in C$ has an address $\omega=\omega(p)$ : its first $n$ symbols $\alpha=\omega \mid n$ are specified by the interval $C_{\alpha}$ of $C^{n}$ in which $p$ lies. A second point $q$ has a different address, since there is some $n$ for which $p$ and $q$ lie in distinct intervals $C_{\alpha}$ and $C_{\beta}$ of $C^{n}$.

In sum, the Cantor set is in one-to-one correspondence with the set $\Omega$ of addresses. Each address $\omega \in \Omega$ defines a point $p(\omega) \in C$ and each point $p \in C$ has a unique address $\omega(p)$. The set $\Omega$ is uncountable. In fact it corresponds bijectively to $\mathbb{R}$. See Exercise 112.

If $S \subset M$ and $\bar{S}=M$ then $S$ is dense in $M$. For example, $\mathbb{Q}$ is dense in $\mathbb{R}$. The set $S$ is somewhere dense if there exists an open nonempty set $U \subset M$ such that $\overline{S \cap U} \supset U$. If $S$ is not somewhere dense then it is nowhere dense.

69 Theorem The Cantor set contains no interval and is nowhere dense in $\mathbb{R}$.
Proof Suppose not and $C$ contains $(a, b)$. Then $(a, b) \subset C^{n}$ for all $n \in \mathbb{N}$. Take $n$ with $1 / 3^{n}<b-a$. Since $(a, b)$ is connected it lies wholly in a single $C^{n}$-interval, say $I$. But $I$ has smaller length than $(a, b)$, which is absurd, so $C$ contains no interval.

Next, suppose $C$ is dense in some nonempty open set $U \subset \mathbb{R}$, i.e., the closure of $C \cap U$ contains $U$. Thus

$$
C=\bar{C} \supset \overline{C \cap U} \supset U \supset(a, b)
$$

contrary to the fact that $C$ contains no interval.

The existence of an uncountable nowhere dense set is astonishing. Even more is true: The Cantor set is a zero set - it has "outer measure zero." By this we mean that, given any $\epsilon>0$, there is a countable covering of $C$ by open intervals $\left(a_{k}, b_{k}\right)$, and the total length of the covering is

$$
\sum_{k=1}^{\infty} b_{k}-a_{k}<\epsilon
$$

(Outer measure is one of the central concepts of Lebesgue Theory. See Chapter 6.) After all, $C$ is a subset of $C^{n}$, which consists of $2^{n}$ closed intervals, each of length $1 / 3^{n}$. If $n$ is large enough then $2^{n} / 3^{n}<\epsilon$. Enlarging each of these closed intervals to an open interval keeps the sum of the lengths $<\epsilon$, and it follows that $C$ is a zero set.

If we discard subintervals of $[0,1]$ in a different way, we can make a fat Cantor set - one that has positive outer measure. Instead of discarding the middle-thirds of intervals at the $n^{\text {th }}$ stage in the construction, we discard only the middle $1 / n$ ! portion. The discards are grossly smaller than the remaining intervals. See Figure 52. The total amount discarded from $[0,1]$ is $<1$, and the total amount remaining, the outer measure of the fat Cantor set, is positive. See Exercise 3.31.

Figure 52 In forming a fat Cantor set, the gap intervals occupy a progressively smaller proportion of the Cantor set intervals.

## 9* Cantor Set Lore

In this section, we explore some arcane features of Cantor sets.
Although the continuous image of a connected set is connected, the continuous image of a disconnected set may well be connected. Just crush the disconnected set to a single point. Nevertheless, I hope you find the following result striking, for it means that the Cantor set $C$ is the universal compact metric space, of which all others are merely shadows.

70 Cantor Surjection Theorem Given a compact nonempty metric space $M$, there is a continuous surjection of $C$ onto $M$.

See Figure 53. Exercise 114 suggests a direct construction of a continuous surjection $C \rightarrow[0,1]$, which is already an interesting fact. The proof of Theorem 70


Figure $53 \sigma$ surjects $C$ onto $M$.
involves a careful use of the address notation from Section 8 and the following simple lemma about dividing a compact metric space $M$ into small pieces. A piece of $M$ is any compact nonempty subset of $M$.

71 Lemma If $M$ is a nonempty compact metric space and $\epsilon>0$ is given then $M$ can be expressed as the finite union of pieces, each of diameter $\leq \epsilon$.

Proof Reduce the covering $\left\{M_{\epsilon / 2}(x): x \in M\right\}$ of $M$ to a finite subcovering and take the closure of each member of the subcovering.

We say that $M$ divides into these small pieces. The metaphor is imperfect because the pieces may overlap. The strategy of the proof of Theorem 70 is to divide $M$ into large pieces, divide the large pieces into small pieces, divide the small pieces into smaller pieces and continue indefinitely. Labeling the pieces coherently with words in two letters leads to the Cantor surjection.

Let $W(n)$ be the set of words in two letters, say $a$ and $b$, having length $n$. Then $\# W(n)=2^{n}$. For example $W(2)$ consists of the four words $a a, b b, a b$, and $b a$.

Using Lemma 71 we divide $M$ into a finite number of pieces of diameter $\leq 1$ and we denote by $\mathcal{M}_{1}$ the collection of these pieces. We choose $n_{1}$ with $2^{n_{1}} \geq \# \mathcal{N}_{1}$ and choose any surjection $w_{1}: W\left(n_{1}\right) \rightarrow \mathcal{N}_{1}$. Since there are enough words in $W\left(n_{1}\right), w_{1}$ exists. We say $w_{1}$ labels $\mathcal{M}_{1}$ and if $w_{1}(\alpha)=L$ then $\alpha$ is a label of $L$.

Then we divide each $L \in \mathcal{M}_{1}$ into finitely many smaller pieces. Let $\mathcal{M}_{2}(L)$ be the collection of these smaller pieces and let

$$
\mathcal{M}_{2}=\bigcup_{L \in \mathcal{M}_{1}} \mathcal{M}_{2}(L)
$$

Choose $n_{2}$ such that $2^{n_{2}} \geq \max \left\{\# \mathcal{N}_{2}(L): L \in \mathcal{N}_{1}\right\}$ and label $\mathcal{N}_{2}$ with words $\alpha \beta \in W\left(n_{1}+n_{2}\right)$ such that

$$
\text { If } \begin{aligned}
L= & w_{1}(\alpha) \text { then } \alpha \beta \text { labels the pieces } S \in \mathcal{M}_{2}(L) \\
& \text { as } \beta \text { varies in } W\left(n_{2}\right) .
\end{aligned}
$$

This labeling amounts to a surjection $w_{2}: W\left(n_{1}+n_{2}\right) \rightarrow \mathcal{M}_{2}$ that is coherent with $w_{1}$ in the sense that $\beta \mapsto w_{2}(\alpha \beta)$ labels the pieces $S \in w_{1}(\alpha)$. Since there are enough words in $W\left(n_{2}\right)$, $w_{2}$ exists. If there are other labels $\alpha^{\prime}$ of $L \in \mathcal{M}_{1}$ then we get other labels $\alpha^{\prime} \beta^{\prime}$ for the pieces $S \in \mathcal{M}_{2}(L)$. We make no effort to correlate them.

Proceeding by induction we get finer and finer divisions of $M$ coherently labeled with longer and longer words. More precisely there is a sequence of divisions $\left(\mathcal{M}_{k}\right)$ and surjections $w_{k}: W_{k}=W\left(n_{1}+\cdots+n_{k}\right) \rightarrow \mathcal{M}_{k}$ such that
(a) The maximum diameter of the pieces $L \in \mathcal{N}_{k}$ tends to zero as $k \rightarrow \infty$.
(b) $\mathcal{M}_{k+1}$ refines $\mathcal{M}_{k}$ in the sense that each $S \in \mathcal{M}_{k+1}$ is contained in some $L \in \mathcal{N}_{k}$. ("The small pieces $S$ are contained in the large pieces $L$.")
(c) If $L \in \mathcal{M}_{k}$ and $\mathcal{M}_{k+1}(L)$ denotes $\left\{S \in \mathcal{M}_{k+1}: S \subset L\right\}$ then

$$
L=\mathbf{U}_{S \in \mathcal{M}_{k+1}(L)} S
$$

(d) The labelings $w_{k}$ are coherent in the sense that if $w_{k}(\alpha)=L \in \mathcal{N}_{k}$ then $\beta \mapsto w_{k+1}(\alpha \beta)$ labels $\mathcal{M}_{k+1}(L)$ as $\beta$ varies in $W\left(n_{k+1}\right)$.

See Figure 54.
Proof of the Cantor Surjection Theorem We are given a nonempty compact metric space $M$ and we seek a continuous surjection $\sigma: C \rightarrow M$ where $C$ is the standard Cantor set.
$C=\bigcap C^{n}$ where $C^{n}$ is the disjoint union of $2^{n}$ closed intervals of length $1 / 3^{n}$. In Section 8 we labeled these $C^{n}$-intervals with words in the letters 0 and 2 having length $n$. (For instance $C_{220}$ is the left $C^{3}$-interval of $C_{22}=[8 / 9,1]$, namely $C_{220}=$ $[8 / 9,25 / 27]$.) We showed there is a natural bijection between $C$ and the set of all infinite words in the letters 0 and 2 defined by

$$
p=\bigcap_{n \in \mathbb{N}} C_{\omega \mid n}
$$



Figure 54 Coherently labeled successive divisions of $M$. They have $n_{1}=2, n_{2}=1$, and $n_{3}=6$. Note that overlabeling is necessary.

We referred to $\omega=\omega(p)$ as the address of $p$. ( $\omega \mid n$ is the truncation of $\omega$ to its first $n$ letters.) See page 107.

For $k=1,2, \ldots$ let $\mathcal{N}_{k}$ be the fine divisions of $M$ constructed above, coherently labeled by $w_{k}$. They obey (a)-(d). Given $p \in C$ we look at the nested sequence of pieces $L_{k}(p) \in \mathcal{M}_{k}$ such that $L_{k}(p)=w_{k}\left(\omega \mid\left(n_{1}+\cdots+n_{k}\right)\right)$ where $\omega=\omega(p)$. That is, we truncate $\omega(p)$ to its first $n_{1}+\cdots+n_{k}$ letters and look at the piece in $\mathcal{N}_{k}$ with this label. (We replace the letters 0 and 2 with $a$ and $b$.) Then $\left(L_{k}(p)\right)$ is a nested decreasing sequence of nonempty compact sets whose diameters tend to 0 as $k \rightarrow \infty$. Thus $\bigcap L_{k}(p)$ is a well defined point in $M$ and we set

$$
\sigma(p)=\bigcap_{k \in \mathbb{N}} L_{k}(p)
$$

We must show that $\sigma$ is a continuous surjection $C \rightarrow M$. Continuity is simple. If $p, p^{\prime} \in C$ are close together then for large $n$ the first $n$ entries of their addresses are equal. This implies that $\sigma(p)$ and $\sigma\left(p^{\prime}\right)$ belong to a common $L_{k}$ and $k$ is large. Since the diameter of $L_{k}$ tends to 0 as $k \rightarrow \infty$ we get continuity.

Surjectivity is also simple. Each $q \in M$ is the intersection of at least one nested sequence of pieces $L_{k} \in \mathcal{M}_{k}$. For $q$ belongs to some piece $L_{1} \in \mathcal{M}_{1}$, and it also belongs
to some subpiece $L_{2} \in \mathcal{M}_{2}\left(L_{1}\right)$, etc. Coherence of the labeling of the $\mathcal{N}_{k}$ implies that for each nested sequence $\left(L_{k}\right)$ there is an infinite word $\alpha=\alpha_{1} \alpha_{2} \alpha_{3} \ldots$ such that $\alpha_{i} \in W\left(n_{i}\right)$ and $L_{k}=w_{k}\left(\alpha_{1} \ldots \alpha_{m}\right)$ with $m=n_{1}+\cdots+n_{k}$. The point $p \in C$ with address $\alpha$ is sent by $\sigma$ to $q$.

## Peano Curves

72 Theorem There exists a Peano curve, a continuous path in the plane which is space-filling in the sense that its image has nonempty interior. In fact there is a Peano curve whose image is the closed unit disc $B^{2}$.

Proof Let $\sigma: C \rightarrow B^{2}$ be a continuous surjection supplied by Theorem 70. Extend $\sigma$ to a map $\tau:[0,1] \rightarrow B^{2}$ by setting

$$
\tau(x)= \begin{cases}\sigma(x) & \text { if } x \in C \\ (1-t) \sigma(a)+t \sigma(b) & \text { if } x=(1-t) a+t b \in(a, b) \\ & \text { and }(a, b) \text { is a gap interval. }\end{cases}
$$

A gap interval is an interval $(a, b) \subset C^{c}$ such that $a, b \in C$. Because $\sigma$ is continuous, $|\sigma(a)-\sigma(b)| \rightarrow 0$ as $|a-b| \rightarrow 0$. Hence $\tau$ is continuous. Its image includes the disc $B^{2}$ and thus has nonempty interior. In fact the image of $\tau$ is exactly $B^{2}$, since the disc is convex and $\tau$ just extends $\sigma$ via linear interpolation. See Figure 55.

This Peano curve cannot be one-to-one since $C$ is not homeomorphic to $B^{2}$. ( $C$ is disconnected while $B^{2}$ is connected.) In fact no Peano curve $\tau$ can be one-to-one. See Exercise 102.

## Cantor Spaces

We say that $M$ is a Cantor space if, like the standard Cantor set $C$, it is compact, nonempty, perfect, and totally disconnected.

73 Moore-Kline Theorem Every Cantor space $M$ is homeomorphic to the standard middle-thirds Cantor set $C$.

A Cantor piece is a nonempty clopen subset $S$ of a Cantor space $M$. It is easy to see that $S$ is also a Cantor space. See Exercise 100. Since a Cantor space is totally disconnected, each point has a small clopen neighborhood $N$. Thus, a Cantor space can always be divided into two disjoint Cantor pieces, $M=U \sqcup U^{c}$.


Figure 55 Filling in the Cantor surjection $\sigma$ to make a Peano space-filling curve $\tau$

74 Cantor Partition Lemma Given a Cantor space $M$ and $\epsilon>0$, there is a number $N$ such that for each $d \geq N$ there is a partition of $M$ into $d$ Cantor pieces of diameter $\leq \epsilon$. (We care most about dyadic d.)

Proof A partition of a set is a division of it into disjoint subsets. In this case the small Cantor pieces form a partition of the Cantor space $M$. Since $M$ is totally disconnected and compact, we can cover it with finitely many clopen neighborhoods $U_{1}, \ldots, U_{m}$ having diameter $\leq \epsilon$. To make the sets $U_{i}$ disjoint, define

$$
\begin{aligned}
V_{1} & =U_{1} \\
V_{2} & =U_{1} \backslash U_{2} \\
& \cdots \\
V_{m} & =U_{m} \backslash\left(U_{1} \cup \cdots \cup U_{m-1}\right)
\end{aligned}
$$

If any $V_{i}$ is empty, discard it. This gives a partition $M=X_{1} \sqcup \cdots \sqcup X_{N}$ into $N \leq m$ Cantor pieces of diameter $\leq \epsilon$.

If $d=N$ this finishes the proof. If $d>N$ then we inductively divide $X_{N}$ into two, and then three, and eventually $d-N+1$ disjoint Cantor pieces; say

$$
X_{N}=Y_{1} \sqcup \cdots \sqcup Y_{d-N+1}
$$

The partition $M=X_{1} \sqcup \cdots \sqcup X_{N-1} \sqcup Y_{1} \sqcup \cdots \sqcup Y_{d-N+1}$ finishes the proof.
Proof of the Moore-Kline Theorem We are given a Cantor space $M$ and we seek a homeomorphism from the standard Cantor set $C$ onto $M$.

By Lemma 74 there is a partition $\mathcal{M}_{1}$ of $M$ into $d_{1}$ nonempty Cantor pieces where $d_{1}=2^{n_{1}}$ is dyadic and the pieces have diameter $\leq 1$. Thus there is a bijection $w_{1}: W_{1} \rightarrow \mathcal{M}_{1}$ where $W_{1}=W\left(n_{1}\right)$.

According to the same lemma, each $L \in \mathcal{M}_{1}$ can be partitioned into $N(L)$ Cantor pieces of diameter $\leq 1 / 2$. Choose a dyadic number

$$
d_{2}=2^{n_{2}} \geq \max \left\{N(L): L \in \mathcal{M}_{1}\right\}
$$

and use the lemma again to partition each $L$ into $d_{2}$ smaller Cantor pieces. These pieces constitute $\mathcal{M}_{2}(L)$, and we set $\mathcal{M}_{2}=\mathbf{U}_{L} \mathcal{M}_{2}(L)$. It is a partition of $M$ having cardinality $d_{1} d_{2}$ and in the natural way described in the proof of Theorem 70 it is coherently labeled by $W_{2}=W\left(n_{1}+n_{2}\right)$. Specifically, for each $L \in \mathcal{M}_{1}$ there is a bijection $w_{L}: W\left(n_{2}\right) \rightarrow \mathcal{M}_{2}(L)$ and we define $w_{2}: W_{2} \rightarrow \mathcal{M}_{2}$ by $w_{2}(\alpha \beta)=S \in \mathcal{M}_{2}$ if and only if $w_{1}(\alpha)=L$ and $w_{L}(\beta)=S$. This $w_{2}$ is a bijection.

Proceeding in exactly the same way, we pass from 2 to 3 , from 3 to 4 , and eventually from $k$ to $k+1$, successively refining the partitions and extending the bijective labelings.

The Cantor surjection constructed in the proof of Theorem 70 is

$$
\sigma(p)=\bigcap_{k} L_{k}(p)
$$

where $L_{k}(p) \in \mathcal{M}_{k}$ has label $\omega(p) \mid m$ with $m=n_{1}+\cdots+n_{k}$. Distinct points $p, p^{\prime} \in C$ have distinct addresses $\omega, \omega^{\prime}$. Because the labelings $w_{k}$ are bijections and the divisions $\mathcal{M}_{k}$ are partitions, $\omega \neq \omega^{\prime}$ implies that for some $k, L_{k}(p) \neq L_{k}\left(p^{\prime}\right)$, and thus $\sigma(p) \neq$ $\sigma\left(p^{\prime}\right)$. That is, $\sigma$ is a continuous bijection $C \rightarrow M$. A continuous bijection from one compact to another is a homeomorphism.

75 Corollary Every two Cantor spaces are homeomorphic.
Proof Immediate from the Moore-Kline Theorem: Each is homeomorphic to $C$.

76 Corollary The fat Cantor set is homeomorphic to the standard Cantor set.
Proof Immediate from the Moore-Kline Theorem.
77 Corollary $A$ Cantor set is homeomorphic to its own Cartesian square; that is, $C \cong C \times C$.

Proof It is enough to check that $C \times C$ is a Cantor space. It is. See Exercise 99.

The fact that a nontrivial space is homeomorphic to its own Cartesian square is disturbing, is it not?

## Ambient Topological Equivalence

Although all Cantor spaces are homeomorphic to each other when considered as abstract metric spaces, they can present themselves in very different ways as subsets of Euclidean space. Two sets $A, B$ in $\mathbb{R}^{m}$ are ambiently homeomorphic if there is a homeomorphism of $\mathbb{R}^{m}$ to itself that sends $A$ onto $B$. For example, the sets

$$
A=\{0\} \cup[1,2] \cup\{3\} \quad \text { and } \quad B=\{0\} \cup\{1\} \cup[2,3]
$$

are homeomorphic when considered as metric spaces, but there is no ambient homeomorphism of $\mathbb{R}$ that carries $A$ to $B$. Similarly, the trefoil knot in $\mathbb{R}^{3}$ is homeomorphic but not ambiently homeomorphic in $\mathbb{R}^{3}$ to a planar circle. See also Exercise 105.

78 Theorem Every two Cantor spaces in $\mathbb{R}$ are ambiently homeomorphic.

Let $M$ be a Cantor space contained in $\mathbb{R}$. According to Theorem 73, $M$ is homeomorphic to the standard Cantor set $C$. We want to find a homeomorphism of $\mathbb{R}$ to itself that carries $C$ to $M$.

The convex hull of $S \subset \mathbb{R}^{m}$ is the smallest convex set $H$ that contains $S$. When $m=1, H$ is the smallest interval that contains $S$.

79 Lemma $A$ Cantor space $M \subset \mathbb{R}$ can be divided into two Cantor pieces whose convex hulls are disjoint.

Proof Obvious from one-dimensionality of $\mathbb{R}$ : Choose a point $x \in \mathbb{R} \backslash M$ such that some points of $M$ lie to the left of $x$ and others lie to its right. Then

$$
M=M \cap(-\infty, x) \sqcup(x, \infty) \cap M
$$

divides $M$ into disjoint Cantor pieces whose convex hulls are disjoint closed intervals. $\square$

Proof of Theorem 78 Let $M \subset \mathbb{R}$ be a Cantor space. We will find a homeomorphism $\tau: \mathbb{R} \rightarrow \mathbb{R}$ sending $C$ to $M$. Lemma 79 leads to Cantor divisions $\mathcal{M}_{k}$ such that the convex hulls of the pieces in each $\mathcal{M}_{k}$ are disjoint. With respect to the left/right order of $\mathbb{R}$, label these pieces in the same way that the Cantor middle third intervals are labeled: $L_{0}$ and $L_{2}$ in $\mathcal{M}_{1}$ are the left and right pieces of $M, L_{00}$ and $L_{02}$ are the left and right pieces of $L_{0}$, and so on. Then the homeomorphism $\sigma: C \rightarrow M$ constructed in Theorems 70 and 73 is automatically monotone increasing. Extend $\sigma$ across the gap intervals affinely as was done in the proof of Theorem 72, and extend it to $\mathbb{R} \backslash[0,1]$ in any affine increasing fashion such that $\tau(0)=\sigma(0)$ and $\tau(1)=\sigma(1)$. Then $\tau: \mathbb{R} \rightarrow \mathbb{R}$ extends $\sigma$ to $\mathbb{R}$. The monotonicity of $\sigma$ implies that $\tau$ is one-to-one, while the continuity of $\sigma$ implies that $\tau$ is continuous. $\tau: \mathbb{R} \rightarrow \mathbb{R}$ is a homeomorphism that carries $C$ onto $M$.

If $M^{\prime} \subset \mathbb{R}$ is a second Cantor space and $\tau^{\prime}: \mathbb{R} \rightarrow \mathbb{R}$ is a homeomorphism that sends $C$ onto $M^{\prime}$ then $\tau^{\prime} \circ \tau^{-1}$ is a homeomorphism of $\mathbb{R}$ that sends $M$ onto $M^{\prime}$.

As an example, one may construct a Cantor set in $\mathbb{R}$ by removing from $[0,1]$ its middle third, then removing from each of the remaining intervals nine symmetrically placed subintervals; then removing from each of the remaining twenty intervals, four asymmetrically placed subintervals; and so forth. In the limit (if the lengths of the remaining intervals tend to zero) we get a nonstandard Cantor set $M$. According to Theorem 78, there is a homeomorphism of $\mathbb{R}$ to itself sending the standard Cantor set $C$ onto $M$.

Another example is the fat Cantor set mentioned on page 108. It too is ambiently homeomorphic to $C$.

Theorem Every two Cantor spaces in $\mathbb{R}^{2}$ are ambiently homeomorphic.
We do not prove this theorem here. The key step is to show $M$ has a dyadic disc partition. That is, $M$ can be divided into a dyadic number of Cantor pieces, each piece contained in the interior of a small topological disc $D_{i}$, the $D_{i}$ being mutually disjoint. (A topological disc is any homeomorph of the closed unit disc $B^{2}$. Smallness refers to diam $D_{i}$.) The proofs I know of the existence of such dyadic partitions are tricky cut-and-paste arguments and are beyond the scope of this book. See Moise's book, Geometric Topology in Dimensions 2 and 3 and also Exercise 138.

## Antoine's Necklace

A Cantor space $M \subset \mathbb{R}^{m}$ is tame if there is an ambient homeomorphism $h:$ $\mathbb{R}^{m} \rightarrow \mathbb{R}^{m}$ that carries the standard Cantor set $C$ (imagined to lie on the $x_{1}$-axis
in $\mathbb{R}^{m}$ ) onto $M$. If $M$ is not tame it is wild. Cantor spaces contained in the line or plane are tame. In 3 -space, however, there are wild ones, Cantor sets $A$ so badly embedded in $\mathbb{R}^{3}$ that they act like curves. It is the lack of a "ball dyadic partition lemma" that causes the problem.

The first wild Cantor set was discovered by Louis Antoine, and is known as Antoine's Necklace. The construction involves the solid torus or anchor ring, which is homeomorphic to the Cartesian product $B^{2} \times S^{1}$. It is easy to imagine a necklace of solid tori: Take an ordinary steel chain and modify it so its first and last links are also linked. See Figure 56.


Figure 56 A necklace of twenty solid tori
Antoine's construction then goes like this. Draw a solid torus $A^{0}$. Interior to $A^{0}$, draw a necklace $A^{1}$ of several small solid tori, and make the necklace encircle the hole of $A^{0}$. Repeat the construction on each solid torus $T$ comprising $A^{1}$. That is, interior to each $T$, draw a necklace of very small solid tori so that it encircles the hole of $T$. The result is a set $A^{2} \subset A^{1}$ which is a necklace of necklaces. In Figure $56, A^{2}$ would consist of 400 solid tori. Continue indefinitely, producing a nested decreasing sequence $A^{0} \supset A^{1} \supset A^{2} \supset \ldots$. The set $A^{n}$ is compact and consists of a large number $\left(20^{n}\right)$ of extremely small solid tori arranged in a hierarchy of necklaces. It is an $n^{\text {th }}$ order necklace. The intersection $A=\bigcap A^{n}$ is a Cantor space, since it is
compact, perfect, nonempty, and totally disconnected. It is homeomorphic to $C$. See Exercise 139.

Certainly A is bizarre, but is it wild? Is there no ambient homeomorphism $h$ of $\mathbb{R}^{3}$ that sends the standard Cantor set $C$ onto $A$ ? The reason that $h$ cannot exist is explained next.


Figure $57 \kappa$ loops through $A^{0}$, which contains the necklace of solid tori.
Referring to Figure 57, the loop $\kappa$ passing through the hole of $A^{0}$ cannot be continuously shrunk to a point in $\mathbb{R}^{3}$ without hitting $A$. For if such a motion of $\kappa$ avoids $A$ then, by compactness, it also avoids one of the high-order necklaces $A^{n}$. In $\mathbb{R}^{3}$ it is impossible to continuously de-link two linked loops, and it is also impossible to continuously de-link a loop from a necklace of loops. (These facts are intuitively believable but hard to prove. See Dale Rolfsen's book, Knots and Links.)

On the other hand, each loop $\lambda$ in $\mathbb{R}^{3} \backslash C$ can be continuously shrunk to a point without hitting $C$. For there is no obstruction to pushing $\lambda$ through the gap intervals of $C$.

Now suppose that there is an ambient homeomorphism $h$ of $\mathbb{R}^{3}$ that sends $C$ to $A$. Then $\lambda=h^{-1}(\kappa)$ is a loop in $\mathbb{R}^{3} \backslash C$, and it can be shrunk to a point in $\mathbb{R}^{3} \backslash C$, avoiding $C$. Applying $h$ to this motion of $\lambda$ continuously shrinks $\kappa$ to a point, avoiding $A$, which we have indicated is impossible. Hence $h$ cannot exist, and $A$ is wild.

## 10* Completion

Many metric spaces are complete (for example, every closed subset of Euclidean space is complete), and completeness is a reasonable property to require of a metric space, especially in light of the following theorem.

80 Completion Theorem Every metric space can be completed.

This means that just as $\mathbb{R}$ completes $\mathbb{Q}$, we can take any metric space $M$ and find a complete metric space $\widehat{M}$ containing $M$ whose metric extends the metric of $M$. To put it another way, $M$ is always a metric subspace of a complete metric space. In a natural sense the completion is uniquely determined by $M$.

81 Lemma Given four points $p, q, x, y \in M$, we have

$$
|d(p, q)-d(x, y)| \leq d(p, x)+d(q, y)
$$

Proof The triangle inequality implies that

$$
\begin{aligned}
& d(x, y) \leq d(x, p)+d(p, q)+d(q, y) \\
& d(p, q) \leq d(p, x)+d(x, y)+d(y, q)
\end{aligned}
$$

and hence

$$
-(d(p, x)+d(q, y)) \leq d(p, q)-d(x, y) \leq(d(p, x)+d(q, y))
$$

A number sandwiched between $-k$ and $k$ has magnitude $\leq k$, which completes the proof.

Proof of the Completion Theorem 80 We consider the collection $\mathcal{C}$ of all Cauchy sequences in $M$, convergent or not, and convert it into the completion of $M$. (This is a bold idea, is it not?) Cauchy sequences $\left(p_{n}\right)$ and $\left(q_{n}\right)$, are co-Cauchy if $d\left(p_{n}, q_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$. Co-Cauchyness is an equivalence relation on $\mathcal{C}$. (This is easy to check.)

Define $\widehat{M}$ to be $\mathcal{C}$ modulo the equivalence relation of being co-Cauchy. Points of $\widehat{M}$ are equivalence classes $P=\left[\left(p_{n}\right)\right]$ such that $\left(p_{n}\right)$ is a Cauchy sequence in $M$. The metric on $\widehat{M}$ is

$$
D(P, Q)=\lim _{n \rightarrow \infty} d\left(p_{n}, q_{n}\right)
$$

where $P=\left[\left(p_{n}\right)\right]$ and $Q=\left[\left(q_{n}\right)\right]$. It only remains to verify three things:
(a) $D$ is a well defined metric on $\widehat{M}$.
(b) $M \subset \widehat{M}$.
(c) $\widehat{M}$ is complete.

None of these assertions is really hard to prove, although the details are somewhat messy because of possible equivalence class/representative ambiguity.
(a) By Lemma 81

$$
\left|d\left(p_{m}, q_{m}\right)-d\left(p_{n}, q_{n}\right)\right| \leq d\left(p_{m}, p_{n}\right)+d\left(q_{m}, q_{n}\right)
$$

Thus $\left(d\left(p_{n}, q_{n}\right)\right)$ is a Cauchy sequence in $\mathbb{R}$, and because $\mathbb{R}$ is complete,

$$
L=\lim _{n \rightarrow \infty} d\left(p_{n}, q_{n}\right)
$$

exists. Let $\left(p_{n}^{\prime}\right)$ and $\left(q_{n}^{\prime}\right)$ be sequences that are co-Cauchy with $\left(p_{n}\right)$ and $\left(q_{n}\right)$, and let

$$
L^{\prime}=\lim _{n \rightarrow \infty} d\left(p_{n}^{\prime}, q_{n}^{\prime}\right)
$$

Then

$$
\left|L-L^{\prime}\right| \leq\left|L-d\left(p_{n}, q_{n}\right)\right|+\left|d\left(p_{n}, q_{n}\right)-d\left(p_{n}^{\prime}, q_{n}^{\prime}\right)\right|+\left|d\left(p_{n}^{\prime}, q_{n}^{\prime}\right)-L^{\prime}\right|
$$

As $n \rightarrow \infty$, the first and third terms tend to 0 . By Lemma 81 , the middle term is

$$
\left|d\left(p_{n}, q_{n}\right)-d\left(p_{n}^{\prime}, q_{n}^{\prime}\right)\right| \leq d\left(p_{n}, p_{n}^{\prime}\right)+d\left(q_{n}, q_{n}^{\prime}\right)
$$

which also tends to 0 as $n \rightarrow \infty$. Hence $L=L^{\prime}$ and $D$ is well defined on $\widehat{M}$. The $d$-distance on $M$ is symmetric and satisfies the triangle inequality. Taking limits, these properties carry over to $D$ on $\widehat{M}$, while positive definiteness follows directly from the co-Cauchy definition.
(b) Think of each $p \in M$ as a constant sequence, $\bar{p}=(p, p, p, p, \ldots)$. Clearly it is Cauchy and clearly the $D$-distance between two constant sequences $\bar{p}$ and $\bar{q}$ is the same as the $d$-distance between the points $p$ and $q$. In this way M is naturally a metric subspace of $\widehat{M}$.
(c) Let $\left(P_{k}\right)_{k \in \mathbb{N}}$ be a Cauchy sequence in $\widehat{M}$. We must find $Q \in \widehat{M}$ to which $P_{k}$ converges as $k \rightarrow \infty$. (Note that $\left(P_{k}\right)$ is a sequence of equivalence classes, not a sequence of points in $M$, and convergence refers to $D$ not $d$.) Because $D$ is well defined we can use a trick to shorten the proof. Observe that every subsequence of a Cauchy sequence is Cauchy, and it and the mother sequence are co-Cauchy. For all the terms far along in the subsequence are also far along in the mother sequence. This lets us take a representative of $P_{k}$ all of whose terms are at distance $<1 / k$ from each other. Call this sequence $\left(p_{k, n}\right)_{n \in \mathbb{N}}$. We have $\left[\left(p_{k, n}\right)\right]=P_{k}$.

Set $q_{n}=p_{n, n}$. We claim that $\left(q_{n}\right)$ is Cauchy and $D\left(P_{k}, Q\right) \rightarrow 0$ as $k \rightarrow \infty$, where $Q=\left[\left(q_{n}\right)\right]$. That is, $\widehat{M}$ is complete.

Let $\epsilon>0$ be given. There exists $N \geq 3 / \epsilon$ such that if $k, \ell \geq N$ then

$$
D\left(P_{k}, P_{\ell}\right) \leq \frac{\epsilon}{3}
$$

and

$$
\begin{aligned}
d\left(q_{k}, q_{\ell}\right) & =d\left(p_{k, k}, p_{\ell, \ell}\right) \\
& \leq d\left(p_{k, k}, p_{k, n}\right)+d\left(p_{k, n}, p_{\ell, n}\right)+d\left(p_{\ell, n}, p_{\ell, \ell}\right) \\
& \leq \frac{1}{k}+d\left(p_{k, n}, p_{\ell, n}\right)+\frac{1}{\ell} \\
& \leq \frac{2 \epsilon}{3}+d\left(p_{k, n}, p_{\ell, n}\right)
\end{aligned}
$$

The inequality is valid for all $n$ and the left-hand side, $d\left(q_{k}, q_{\ell}\right)$, does not depend on $n$. The limit of $d\left(p_{k, n}, p_{\ell, n}\right)$ as $n \rightarrow \infty$ is $D\left(P_{k}, P_{\ell}\right)$, which we know to be $<\epsilon / 3$. Thus, if $k, \ell \geq N$ then $d\left(q_{k}, q_{\ell}\right)<\epsilon$ and $\left(q_{n}\right)$ is Cauchy. Similarly we see that $P_{k} \rightarrow Q$ as $k \rightarrow \infty$. For, given $\epsilon>0$, we choose $N \geq 2 / \epsilon$ such that if $k, n \geq N$ then $d\left(q_{k}, q_{n}\right)<\epsilon / 2$, from which it follows that

$$
\begin{aligned}
d\left(p_{k, n}, q_{n}\right) & \leq d\left(p_{k, n}, p_{k, k}\right)+d\left(p_{k, k}, q_{n}\right) \\
& =d\left(p_{k, n}, p_{k, k}\right)+d\left(q_{k}, q_{n}\right) \\
& \leq \frac{1}{k}+\frac{\epsilon}{2}<\epsilon .
\end{aligned}
$$

The limit of the left-hand side of this inequality, as $n \rightarrow \infty$, is $D\left(P_{k}, Q\right)$. Thus

$$
\lim _{k \rightarrow \infty} P_{k}=Q
$$

and $\widehat{M}$ is complete.

Uniqueness of the completion is not surprising, and is left as Exercise 106. A different proof of the Completion Theorem is sketched in Exercise 4.39.

## A Second Construction of $\mathbb{R}$ from $\mathbb{Q}$

In the particular case that the metric space $M$ is $\mathbb{Q}$, the Completion Theorem leads to a construction of $\mathbb{R}$ from $\mathbb{Q}$ via Cauchy sequences. Note, however, that applying the theorem as it stands involves circular reasoning, for its proof uses completeness of $\mathbb{R}$ to define the metric $D$. Instead, we use only the Cauchy sequence strategy.

Convergence and Cauchyness for sequences of rational numbers are concepts that make perfect sense without a priori knowledge of $\mathbb{R}$. Just take all epsilons and deltas
in the definitions to be rational. The Cauchy completion $\widehat{\mathbb{Q}}$ of $\mathbb{Q}$ is the collection $\mathcal{C}$ of Cauchy sequences in $\mathbb{Q}$ modulo the equivalence relation of being co-Cauchy.

We claim that $\widehat{\mathbb{Q}}$ is a complete ordered field. That is, $\widehat{\mathbb{Q}}$ is just another version of $\mathbb{R}$. The arithmetic on $\widehat{\mathbb{Q}}$ is defined by

$$
\begin{aligned}
P+Q & =\left[\left(p_{n}+q_{n}\right)\right] & P-Q=\left[\left(p_{n}-q_{n}\right)\right] \\
P Q & =\left[\left(p_{n} q_{n}\right)\right] & P / Q=\left[\left(p_{n} / q_{n}\right)\right]
\end{aligned}
$$

where $P=\left[\left(p_{n}\right)\right]$ and $Q=\left[\left(q_{n}\right)\right]$. Of course $Q \neq[(0,0, \ldots)]$ in the fraction $P / Q$. Exercise 134 asks you to check that these natural definitions make $\widehat{\mathbb{Q}}$ a field. Although there are many things to check - well definedness, commutativity, and so forth - all are effortless. There are no sixteen case proofs as with cuts. Also, just as with metric spaces, $\mathbb{Q}$ is naturally a subfield of $\widehat{\mathbb{Q}}$ when we think of $r \in \mathbb{Q}$ as the constant sequence $\bar{r}=[(r, r, \ldots)]$.

That's the easy part - now the rest.
To define the order relation on $\widehat{\mathbb{Q}}$ we rework some of the cut ideas. If $P \in \widehat{\mathbb{Q}}$ has a representative $\left[\left(p_{n}\right)\right]$, such that for some $\epsilon>0$, we have $p_{n} \geq \epsilon$ for all $n$ then $P$ is positive. If $-P$ is positive then $P$ is negative.

Then we define $P \prec Q$ if $Q-P$ is positive. Exercise 135 asks you to check that this defines an order on $\widehat{\mathbb{Q}}$, consistent with the standard order $<$ on $\mathbb{Q}$ in the sense that for all $p, q \in \mathbb{Q}$ we have $p<q \Longleftrightarrow \bar{p} \prec \bar{q}$. In particular, you are asked to prove the trichotomy property: Each $P \in \widehat{\mathbb{Q}}$ is either positive, negative, or zero, and these possibilities are mutually exclusive.

Combining Cauchyness with the definition of $\prec$ gives

$$
\begin{align*}
& P=\left[\left(p_{n}\right)\right] \prec Q=\left[\left(q_{n}\right)\right] \quad \Longleftrightarrow \quad \text { there exist } \epsilon>0 \text { and } N \in \mathbb{N} \\
& \text { such that for all } m, n \geq N,  \tag{1}\\
& \text { we have } p_{m}+\epsilon<q_{n} .
\end{align*}
$$

It remains to check the least upper bound property. Let $\mathcal{P}$ be a nonempty subset of $\widehat{\mathbb{Q}}$ that is bounded above. We must find a least upper bound for $\mathcal{P}$.

First of all, since $\mathcal{P}$ is bounded there is a $B=\left(b_{n}\right) \in \widehat{\mathbb{Q}}$ such that $P \prec B$ for all $P \in \mathcal{P}$. We can choose $B$ so its terms lie at distance $\leq 1$ from each other. Set $b=b_{1}+1$. Then $\bar{b}$ is an upper bound for $\mathcal{P}$. Since $\mathbb{Q}$ is Archimedean there is an integer $m \geq b$, and $\bar{m}$ is also an upper bound for $\mathcal{P}$. By the same reasoning $\mathcal{P}$ has upper bounds $\bar{r}$ such that $r$ is a dyadic fraction with arbitrarily large denominator $2^{n}$.

Since $\mathcal{P}$ is nonempty, the same reasoning shows that there are dyadic fractions $s$ with large denominators such that $\bar{s}$ is not an upper bound for $\mathcal{P}$.

We assert that the least upper bound for $\mathcal{P}$ is the equivalence class $Q$ of the following Cauchy sequence $\left(q_{0}, q_{1}, q_{2}, \ldots\right)$.
(a) $q_{0}$ is the smallest integer such that $\overline{q_{0}}$ is an upper bound for $\mathcal{P}$.
(b) $q_{1}$ is the smallest fraction with denominator 2 such that $\overline{q_{1}}$ is an upper bound for $\mathcal{P}$.
(c) $q_{2}$ is the smallest fraction with denominator 4 such that $\overline{q_{2}}$ is an upper bound for $\mathcal{P}$.
(d) $\ldots$
(e) $q_{n}$ is the smallest fraction with denominator $2^{n}$ such that $\overline{q_{n}}$ is an upper bound for $\mathcal{P}$.

The sequence $\left(q_{n}\right)$ is well defined because some but not all dyadic fractions with denominator $2^{n}$ are upper bounds for $\mathcal{P}$. By construction $\left(q_{n}\right)$ is monotone decreasing and $q_{n-1}-q_{n} \leq 1 / 2^{n}$. Thus, if $m \leq n$ then

$$
\begin{gathered}
0 \leq q_{m}-q_{n}=q_{m}-q_{m+1}+q_{m+1}-q_{m+2}+\cdots+q_{n-1}-q_{n} \\
\leq \frac{1}{2^{m+1}}+\cdots+\frac{1}{2^{n}}<\frac{1}{2^{m}}
\end{gathered}
$$

It follows that $\left(q_{n}\right)$ is Cauchy and $Q=\left[\left(q_{n}\right)\right] \in \widehat{\mathbb{Q}}$.
Suppose that $Q$ is not an upper bound for $\mathcal{P}$. Then there is some $P=\left[\left(p_{n}\right)\right] \in \mathcal{P}$ with $Q \prec P$. By (1), there is an $\epsilon>0$ and an $N$ such that for all $n \geq N$,

$$
q_{N}+\epsilon<p_{n}
$$

It follows that $\overline{q_{N}} \prec P$, a contradiction to $\overline{q_{N}}$ being an upper bound for $\mathcal{P}$.
On the other hand suppose there is a smaller upper bound for $\mathcal{P}$, say $R=\left(r_{n}\right) \prec$ $Q$. By (1) there are $\epsilon>0$ and $N$ such that for all $m, n \geq N$,

$$
r_{m}+\epsilon<q_{n}
$$

Fix a $k \geq N$ with $1 / 2^{k}<\epsilon$. Then for all $m \geq N$,

$$
r_{m}<q_{k}-\epsilon<q_{k}-\frac{1}{2^{k}}
$$

By (1), $R \prec \overline{q_{k}-1 / 2^{k}}$. Since $R$ is an upper bound for $\mathcal{P}$, so is $\overline{q_{k}-1 / 2^{k}}$, a contradiction to $q_{k}$ being the smallest fraction with denominator $2^{k}$ such that $\overline{q_{k}}$ is an upper bound for $\mathcal{P}$. Therefore, $Q$ is indeed a least upper bound for $\mathcal{P}$.

This completes the verification that the Cauchy completion of $\mathbb{Q}$ is a complete ordered field. Uniqueness implies that it is isomorphic to the complete ordered field $\mathbb{R}$ constructed by means of Dedekind cuts in Section 2 of Chapter 1. Decide for yourself which of the two constructions of the real number system you like better - cuts or Cauchy sequences. Cuts make least upper bounds straightforward and algebra awkward, while with Cauchy sequences it is the reverse.

## Exercises

1. An ant walks on the floor, ceiling, and walls of a cubical room. What metric is natural for the ant's view of its world? What metric would a spider consider natural? If the ant wants to walk from a point $p$ to a point $q$, how could it determine the shortest path?
2. Why is the sum metric on $\mathbb{R}^{2}$ called the Manhattan metric and the taxicab metric?
3. What is the set of points in $\mathbb{R}^{3}$ at distance exactly $1 / 2$ from the unit circle $S^{1}$ in the plane,

$$
\begin{aligned}
& T=\left\{p \in \mathbb{R}^{3}: \exists q \in S^{1} \text { and } d(p, q)=1 / 2\right. \\
&\text { and for all } \left.q^{\prime} \in S^{1} \text { we have } d(p, q) \leq d\left(p, q^{\prime}\right)\right\} ?
\end{aligned}
$$

4. Write out a proof that the discrete metric on a set $M$ is actually a metric.

5 . For $p, q \in S^{1}$, the unit circle in the plane, let

$$
d_{a}(p, q)=\min \{|\measuredangle(p)-\measuredangle(q)|, 2 \pi-|\measuredangle(p)-\measuredangle(q)|\}
$$

where $\measuredangle(z) \in[0,2 \pi)$ refers to the angle that $z$ makes with the positive $x$-axis. Use your geometric talent to prove that $d_{a}$ is a metric on $S^{1}$.
6 . For $p, q \in[0, \pi / 2)$ let

$$
d_{s}(p, q)=\sin |p-q| .
$$

Use your calculus talent to decide whether $d_{s}$ is a metric.
7. Prove that every convergent sequence $\left(p_{n}\right)$ in a metric space $M$ is bounded, i.e., that for some $r>0$, some $q \in M$, and all $n \in \mathbb{N}$, we have $p_{n} \in M_{r} q$.
8. Consider a sequence $\left(x_{n}\right)$ in the metric space $\mathbb{R}$.
(a) If $\left(x_{n}\right)$ converges in $\mathbb{R}$ prove that the sequence of absolute values $\left(\left|x_{n}\right|\right)$ converges in $\mathbb{R}$.
(b) State the converse.
(c) Prove or disprove it.
9. A sequence $\left(x_{n}\right)$ in $\mathbb{R}$ increases if $n<m$ implies $x_{n} \leq x_{m}$. It strictly increases if $n<m$ implies $x_{n}<x_{m}$. It decreases or strictly decreases if $n<m$ always implies $x_{n} \geq x_{m}$ or always implies $x_{n}>x_{m}$. A sequence is monotone if it increases or it decreases. Prove that every sequence in $\mathbb{R}$ which is monotone and bounded converges in $\mathbb{R} .^{\dagger}$
10. Prove that the least upper bound property is equivalent to the "monotone sequence property" that every bounded monotone sequence converges.

[^8]11. Let $\left(x_{n}\right)$ be a sequence in $\mathbb{R}$.
*(a) Prove that $\left(x_{n}\right)$ has a monotone subsequence.
(b) How can you deduce that every bounded sequence in $\mathbb{R}$ has a convergent subsequence?
(c) Infer that you have a second proof of the Bolzano-Weierstrass Theorem in $\mathbb{R}$.
(d) What about the Heine-Borel Theorem?
12. Let $\left(p_{n}\right)$ be a sequence and $f: \mathbb{N} \rightarrow \mathbb{N}$ be a bijection. The sequence $\left(q_{k}\right)_{k \in \mathbb{N}}$ with $q_{k}=p_{f(k)}$ is a rearrangement of $\left(p_{n}\right)$.
(a) Are limits of a sequence unaffected by rearrangement?
(b) What if $f$ is an injection?
(c) A surjection?
13. Assume that $f: M \rightarrow N$ is a function from one metric space to another which satisfies the following condition: If a sequence $\left(p_{n}\right)$ in $M$ converges then the sequence $\left(f\left(p_{n}\right)\right)$ in $N$ converges. Prove that $f$ is continuous. [This result improves Theorem 4.]
14. The simplest type of mapping from one metric space to another is an isometry. It is a bijection $f: M \rightarrow N$ that preserves distance in the sense that for all $p, q \in M$ we have
$$
d_{N}(f p, f q)=d_{M}(p, q)
$$

If there exists an isometry from $M$ to $N$ then $M$ and $N$ are said to be isometric, $M \equiv N$. You might have two copies of a unit equilateral triangle, one centered at the origin and one centered elsewhere. They are isometric. Isometric metric spaces are indistinguishable as metric spaces.
(a) Prove that every isometry is continuous.
(b) Prove that every isometry is a homeomorphism.
(c) Prove that $[0,1]$ is not isometric to $[0,2]$.
15. Prove that isometry is an equivalence relation: If $M$ is isometric to $N$, show that $N$ is isometric to $M$; show that each $M$ is isometric to itself (what mapping of $M$ to $M$ is an isometry?); if $M$ is isometric to $N$ and $N$ is isometric to $P$, show that $M$ is isometric to $P$.
16. Is the perimeter of a square isometric to the circle? Homeomorphic? Explain.
17. Which capital letters of the Roman alphabet are homeomorphic? Are any isometric? Explain.
18. Is $\mathbb{R}$ homeomorphic to $\mathbb{Q}$ ? Explain.
19. Is $\mathbb{Q}$ homeomorphic to $\mathbb{N}$ ? Explain.
20. What function (given by a formula) is a homeomorphism from $(-1,1)$ to $\mathbb{R}$ ? Is every open interval homeomorphic to $(0,1)$ ? Why or why not?
21. Is the plane minus four points on the $x$-axis homeomorphic to the plane minus four points in an arbitrary configuration?
22. If every closed and bounded subset of a metric space $M$ is compact, does it follow that $M$ is complete? (Proof or counterexample.)
23. $(0,1)$ is an open subset of $\mathbb{R}$ but not of $\mathbb{R}^{2}$, when we think of $\mathbb{R}$ as the $x$-axis in $\mathbb{R}^{2}$. Prove this.
24. For which intervals $[a, b]$ in $\mathbb{R}$ is the intersection $[a, b] \cap \mathbb{Q}$ a clopen subset of the metric space $\mathbb{Q}$ ?
25. Prove directly from the definition of closed set that every singleton subset of a metric space $M$ is a closed subset of $M$. Why does this imply that every finite set of points is also a closed set?
26. Prove that a set $U \subset M$ is open if and only if none of its points are limits of its complement.
27. If $S, T \subset M$, a metric space, and $S \subset T$, prove that
(a) $\bar{S} \subset \bar{T}$.
(b) $\operatorname{int}(S) \subset \operatorname{int}(T)$.
28. A map $f: M \rightarrow N$ is open if for each open set $U \subset M$, the image set $f(U)$ is open in $N$.
(a) If $f$ is open, is it continuous?
(b) If $f$ is a homeomorphism, is it open?
(c) If $f$ is an open, continuous bijection, is it a homeomorphism?
(d) If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous surjection, must it be open?
(e) If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous, open surjection, must it be a homeomorphism?
(f) What happens in (e) if $\mathbb{R}$ is replaced by the unit circle $S^{1}$ ?
29. Let $\mathcal{T}$ be the collection of open subsets of a metric space $M$, and $\mathcal{K}$ the collection of closed subsets. Show that there is a bijection from $\mathcal{T}$ onto $\mathcal{K}$.
30. Consider a two-point set $M=\{a, b\}$ whose topology consists of the two sets, $M$ and the empty set. Why does this topology not arise from a metric on $M$ ?
31. Prove the following.
(a) If $U$ is an open subset of $\mathbb{R}$ then it consists of countably many disjoint intervals $U=】 U_{i}$. (Unbounded intervals $(-\infty, b)$, $(a, \infty)$, and $(-\infty, \infty)$ are permitted.)
(b) Prove that these intervals $U_{i}$ are uniquely determined by $U$. In other words, there is only one way to express $U$ as a disjoint union of open intervals.
(c) If $U, V \subset \mathbb{R}$ are both open, so $U=\bigsqcup U_{i}$ and $V=\bigsqcup V_{j}$ where $U_{i}$ and $V_{j}$ are open intervals, show that $U$ and $V$ are homeomorphic if and only if there are equally many $U_{i}$ and $V_{j}$.
32. Show that every subset of $\mathbb{N}$ is clopen. What does this tell you about every function $f: \mathbb{N} \rightarrow M$, where $M$ is a metric space?
33. (a) Find a metric space in which the boundary of $M_{r} p$ is not equal to the sphere of radius $r$ at $p, \partial\left(M_{r} p\right) \neq\{x \in M: d(x, p)=r\}$.
(b) Need the boundary be contained in the sphere?
34. Use the Inheritance Principle to prove Corollary 15.
35. Prove that $S$ clusters at $p$ if and only if for each $r>0$ there is a point $q \in$ $M_{r} p \cap S$, such that $q \neq p$.
36. Construct a set with exactly three cluster points.
37. Construct a function $f: \mathbb{R} \rightarrow \mathbb{R}$ that is continuous only at points of $\mathbb{Z}$.
38. Let $X, Y$ be metric spaces with metrics $d_{X}, d_{Y}$, and let $M=X \times Y$ be their Cartesian product. Prove that the three natural metrics $d_{E}, d_{\max }$, and $d_{\text {sum }}$ on $M$ are actually metrics. [Hint: Cauchy-Schwarz.]
39. (a) Prove that every convergent sequence is bounded. That is, if $\left(p_{n}\right)$ converges in the metric space $M$, prove that there is some neighborhood $M_{r} q$ containing the set $\left\{p_{n}: n \in \mathbb{N}\right\}$.
(b) Is the same true for a Cauchy sequence in an incomplete metric space?
40. Let $M$ be a metric space with metric $d$. Prove that the following are equivalent.
(a) $M$ is homeomorphic to $M$ equipped with the discrete metric.
(b) Every function $f: M \rightarrow M$ is continuous.
(c) Every bijection $g: M \rightarrow M$ is a homeomorphism.
(d) $M$ has no cluster points.
(e) Every subset of $M$ is clopen.
(f) Every compact subset of $M$ is finite.
41. Let $\left\|\|\right.$ be any norm on $\mathbb{R}^{m}$ and let $B=\left\{x \in \mathbb{R}^{m}:\|x\| \leq 1\right\}$. Prove that $B$ is compact. [Hint: It suffices to show that $B$ is closed and bounded with respect to the Euclidean metric.]
42. What is wrong with the following "proof" of Theorem 28? "Let $\left(\left(a_{n}, b_{n}\right)\right)$ be any sequence in $A \times B$ where $A$ and $B$ are compact. Compactness implies the existence of subsequences $\left(a_{n_{k}}\right)$ and $\left(b_{n_{k}}\right)$ converging to $a \in A$ and $b \in B$ as $k \rightarrow \infty$. Therefore $\left(\left(a_{n_{k}}, b_{n_{k}}\right)\right)$ is a subsequence of $\left(\left(a_{n}, b_{n}\right)\right)$ that converges to a limit in $A \times B$, proving that $A \times B$ is compact."
43. Assume that the Cartesian product of two nonempty sets $A \subset M$ and $B \subset N$ is compact in $M \times N$. Prove that $A$ and $B$ are compact.
44. Consider a function $f: M \rightarrow \mathbb{R}$. Its graph is the set

$$
\{(p, y) \in M \times \mathbb{R}: y=f p\}
$$

(a) Prove that if $f$ is continuous then its graph is closed (as a subset of $M \times \mathbb{R}$ ).
(b) Prove that if $f$ is continuous and $M$ is compact then its graph is compact.
(c) Prove that if the graph of $f$ is compact then $f$ is continuous.
(d) What if the graph is merely closed? Give an example of a discontinuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ whose graph is closed.
45. Draw a Cantor set $C$ on the circle and consider the set $A$ of all chords between points of $C$.
(a) Prove that $A$ is compact.
*(b) Is $A$ convex?
46. Assume that $A, B$ are compact, disjoint, nonempty subsets of $M$. Prove that there are $a_{0} \in A$ and $b_{0} \in B$ such that for all $a \in A$ and $b \in B$ we have

$$
d\left(a_{0}, b_{0}\right) \leq d(a, b)
$$

[The points $a_{0}, b_{0}$ are closest together.]
47. Suppose that $A, B \subset \mathbb{R}^{2}$.
(a) If $A$ and $B$ are homeomorphic, are their complements homeomorphic?
*(b) What if $A$ and $B$ are compact?
***(c) What if $A$ and $B$ are compact and connected?
48. Prove that there is an embedding of the line as a closed subset of the plane, and there is an embedding of the line as a bounded subset of the plane, but there is no embedding of the line as a closed and bounded subset of the plane.
*49. Construct a subset $A \subset \mathbb{R}$ and a continuous bijection $f: A \rightarrow A$ that is not a homeomorphism. [Hint: By Theorem $36 A$ must be noncompact.]
${ }^{* *} 50$. Construct nonhomeomorphic connected, closed subsets $A, B \subset \mathbb{R}^{2}$ for which there exist continuous bijections $f: A \rightarrow B$ and $g: B \rightarrow A$. [Hint: By Theorem $36 A$ and $B$ must be noncompact.]
${ }^{* * *} 51$. Do there exist nonhomeomorphic closed sets $A, B \subset \mathbb{R}$ for which there exist continuous bijections $f: A \rightarrow B$ and $g: B \rightarrow A$ ?
52. Let $\left(A_{n}\right)$ be a nested decreasing sequence of nonempty closed sets in the metric space $M$.
(a) If $M$ is complete and $\operatorname{diam} A_{n} \rightarrow 0$ as $n \rightarrow \infty$, show that $\bigcap A_{n}$ is exactly one point.
(b) To what assertions do the sets $[n, \infty)$ provide counterexamples?
53. Suppose that $\left(K_{n}\right)$ is a nested sequence of compact nonempty sets, $K_{1} \supset K_{2} \supset$ $\ldots$, and $K=\bigcap K_{n}$. If for some $\mu>0, \operatorname{diam} K_{n} \geq \mu$ for all $n$, is it true that $\operatorname{diam} K \geq \mu$ ?
54. If $f: A \rightarrow B$ and $g: C \rightarrow B$ such that $A \subset C$ and for each $a \in A$ we have $f(a)=g(a)$ then $g$ extends $f$. We also say that $f$ extends to $g$. Assume that $f: S \rightarrow \mathbb{R}$ is a uniformly continuous function defined on a subset $S$ of a metric space $M$.
(a) Prove that $f$ extends to a uniformly continuous function $\bar{f}: \bar{S} \rightarrow \mathbb{R}$.
(b) Prove that $\bar{f}$ is the unique continuous extension of $f$ to a function defined on $\bar{S}$.
(c) Prove the same things when $\mathbb{R}$ is replaced with a complete metric space $N$.
55. The distance from a point $p$ in a metric space $M$ to a nonempty subset $S \subset M$ is defined to be dist $(p, S)=\inf \{d(p, s): s \in S\}$.
(a) Show that $p$ is a limit of $S$ if and only if $\operatorname{dist}(p, S)=0$.
(b) Show that $p \mapsto \operatorname{dist}(p, S)$ is a uniformly continuous function of $p \in M$.
56. Prove that the 2 -sphere is not homeomorphic to the plane.
57. If $S$ is connected, is the interior of $S$ connected? Prove this or give a counterexample.
58. Theorem 49 states that the closure of a connected set is connected.
(a) Is the closure of a disconnected set disconnected?
(b) What about the interior of a disconnected set?
*59. Prove that every countable metric space (not empty and not a singleton) is disconnected. [Astonishingly, there exists a countable topological space which is connected. Its topology does not arise from a metric.]
60. (a) Prove that a continuous function $f: M \rightarrow \mathbb{R}$, all of whose values are integers, is constant provided that $M$ is connected.
(b) What if all the values are irrational?
61. Prove that the (double) cone $\left\{(x, y, z) \in \mathbb{R}^{3}: x^{2}+y^{2}=z^{2}\right\}$ is path-connected.
62. Prove that the annulus $A=\left\{z \in \mathbb{R}^{2}: r \leq|z| \leq R\right\}$ is connected.
63. A subset $E$ of $\mathbb{R}^{m}$ is starlike if it contains a point $p_{0}$ (called a center for $E$ ) such that for each $q \in E$, the segment between $p_{0}$ and $q$ lies in $E$.
(a) If $E$ is convex and nonempty prove that it is starlike.
(b) Why is the converse false?
(c) Is every starlike set connected?
(d) Is every connected set starlike? Why or why not?
*64. Suppose that $E \subset \mathbb{R}^{m}$ is open, bounded, and starlike, and $p_{0}$ is a center for $E$.
(a) Is it true or false that all points $p_{1}$ in a small enough neighborhood of $p_{0}$ are also centers for $E$ ?
(b) Is the set of centers convex?
(c) Is it closed as a subset of $E$ ?
(d) Can it consist of a single point?
65. Suppose that $A, B \subset \mathbb{R}^{2}$ are convex, closed, and have nonempty interiors.
(a) Prove that $A, B$ are the closure of their interiors.
(b) If $A, B$ are compact, prove that they are homeomorphic.
[Hint: Draw a picture.]
66. (a) Prove that every connected open subset of $\mathbb{R}^{m}$ is path-connected.
(b) Is the same true for open connected subsets of the circle?
(c) What about connected nonopen subsets of the circle?
67. List the convex subsets of $\mathbb{R}$ up to homeomorphism. How many are there and how many are compact?
68. List the closed convex sets in $\mathbb{R}^{2}$ up to homeomorphism. There are nine. How many are compact?
*69. Generalize Exercises 65 and 68 to $\mathbb{R}^{3}$; to $\mathbb{R}^{m}$.
70. Prove that $(a, b)$ and $[a, b)$ are not homeomorphic metric spaces.
71. Let $M$ and $N$ be nonempty metric spaces.
(a) If $M$ and $N$ are connected prove that $M \times N$ is connected.
(b) What about the converse?
(c) Answer the questions again for path-connectedness.
72. Let $H$ be the hyperbola $\left\{(x, y) \in \mathbb{R}^{2}: x y=1\right.$ and $\left.x, y>0\right\}$ and let $X$ be the $x$-axis.
(a) Is the set $S=X \cup H$ connected?
(b) What if we replace $H$ with the graph $G$ of any continuous positive function $f: \mathbb{R} \rightarrow(0, \infty)$; is $X \cup G$ connected?
(c) What if $f$ is everywhere positive but discontinuous at just one point.
73. Is the disc minus a countable set of points connected? Path-connected? What about the sphere or the torus instead of the disc?
74. Let $S=\mathbb{R}^{2} \backslash \mathbb{Q}^{2}$. (Points $(x, y) \in S$ have at least one irrational coordinate.) Is $S$ connected? Path-connected? Prove or disprove.
*75. An arc is a path with no self-intersection. Define the concept of arc-connectedness and prove that a metric space is path-connected if and only if it is arc-connected.
76. (a) The intersection of connected sets need not be connected. Give an example.
(b) Suppose that $S_{1}, S_{2}, S_{3}, \ldots$ is a sequence of connected, closed subsets of the plane and $S_{1} \supset S_{2} \supset \ldots$ Is $S=\bigcap S_{n}$ connected? Give a proof or counterexample.
*(c) Does the answer change if the sets are compact?
(d) What is the situation for a nested decreasing sequence of compact pathconnected sets?
77. If a metric space $M$ is the union of path-connected sets $S_{\alpha}$, all of which have the nonempty path-connected set $K$ in common, is $M$ path-connected?
78. $\left(p_{1}, \ldots, p_{n}\right)$ is an $\boldsymbol{\epsilon}$-chain in a metric space $M$ if for each $i$ we have $p_{i} \in M$ and $d\left(p_{i}, p_{i+1}\right)<\epsilon$. The metric space is chain-connected if for each $\epsilon>0$ and each pair of points $p, q \in M$ there is an $\epsilon$-chain from $p$ to $q$.
(a) Show that every connected metric space is chain-connected.
(b) Show that if $M$ is compact and chain-connected then it is connected.
(c) Is $\mathbb{R} \backslash \mathbb{Z}$ chain-connected?
(d) If $M$ is complete and chain-connected, is it connected?
79. Prove that if $M$ is nonempty, compact, locally path-connected, and connected then it is path-connected. (See Exercise 143, below.)
80. The Hawaiian earring is the union of circles of radius $1 / n$ and center $x=$ $\pm 1 / n$ on the $x$-axis, for $n \in \mathbb{N}$. See Figure 27 on page 58.
(a) Is it connected?
(b) Path-connected?
(c) Is it homeomorphic to the one-sided Hawaiian earring?
*81. The topologist's sine curve is the set

$$
\{(x, y): x=0 \text { and }|y| \leq 1 \text { or } 0<x \leq 1 \text { and } y=\sin 1 / x\} .
$$

See Figure 43. The topologist's sine circle is shown in Figure 58. (It is the union of a circular arc and the topologist's sine curve.) Prove that it is pathconnected but not locally path-connected. ( $M$ is locally path-connected if for each $p \in M$ and each neighborhood $U$ of $p$ there is a path-connected subneighborhood $V$ of $p$.)


Figure 58 The topologist's sine circle
82. The graph of $f: M \rightarrow \mathbb{R}$ is the set $\{(x, y) \in M \times \mathbb{R}: y=f x\}$.
(a) If $M$ is connected and $f$ is continuous, prove that the graph of $f$ is connected.
(b) Give an example to show that the converse is false.
(c) If $M$ is path-connected and $f$ is continuous, show that the graph is pathconnected.
(d) What about the converse?
83. The open cylinder is $(0,1) \times S^{1}$. The punctured plane is $\mathbb{R}^{2} \backslash\{0\}$.
(a) Prove that the open cylinder is homeomorphic to the punctured plane.
(b) Prove that the open cylinder, the double cone, and the plane are not homeomorphic.
84. Is the closed strip $\left\{(x, y) \in \mathbb{R}^{2}: 0 \leq x \leq 1\right\}$ homeomorphic to the closed half-plane $\left\{(x, y) \in \mathbb{R}^{2}: x \geq 0\right\}$ ? Prove or disprove.
85. Suppose that $M$ is compact and that $\mathcal{U}$ is an open covering of $M$ which is "redundant" in the sense that each $p \in M$ is contained in at least two members of $\mathcal{U}$. Show that $\mathcal{U}$ reduces to a finite subcovering with the same property.
86. Suppose that every open covering of $M$ has a positive Lebesgue number. Give an example of such an $M$ that is not compact.

Exercises 87-94 treat the basic theorems in the chapter, avoiding the use of sequences. The proofs will remain valid in general topological spaces.
87. Give a direct proof that $[a, b]$ is covering compact. [Hint: Let $\mathcal{U}$ be an open covering of $[a, b]$ and consider the set

$$
C=\{x \in[a, b]: \text { finitely many members of } \mathcal{U} \text { cover }[a, x]\} .
$$

Use the least upper bound principle to show that $b \in C$.]
88. Give a direct proof that a closed subset $A$ of a covering compact set $K$ is covering compact. [Hint: If $\mathcal{U}$ is an open covering of $A$, adjoin the set $W=M \backslash A$ to $\mathcal{U}$. Is $\mathcal{W}=\mathcal{U} \cup\{W\}$ an open covering of $K$ ? If so, so what?]
89. Give a proof of Theorem 36 using open coverings. That is, assume $A$ is a covering compact subset of $M$ and $f: M \rightarrow N$ is continuous. Prove directly that $f A$ is covering compact. [Hint: What is the criterion for continuity in terms of preimages?]
90. Suppose that $f: M \rightarrow N$ is a continuous bijection and $M$ is covering compact. Prove directly that $f$ is a homeomorphism.
91. Suppose that $M$ is covering compact and that $f: M \rightarrow N$ is continuous. Use the Lebesgue number lemma to prove that $f$ is uniformly continuous. [Hint: Consider the covering of $N$ by $\epsilon / 2$-neighborhoods $\left\{N_{\epsilon / 2}(q): q \in N\right\}$ and its preimage in $M,\left\{f^{\text {pre }}\left(N_{\epsilon / 2}(q)\right): q \in N\right\}$.]
92. Give a direct proof that the nested decreasing intersection of nonempty covering compact sets is nonempty. [Hint: If $A_{1} \supset A_{2} \supset \ldots$ are covering compact, consider the open sets $U_{n}=A_{n}^{c}$. If $\bigcap A_{n}=\emptyset$, what does $\left\{U_{n}\right\}$ cover?]
93. Generalize Exercise 92 as follows. Suppose that $M$ is covering compact and $\mathcal{C}$ is a collection of closed subsets of $M$ such that every intersection of finitely many members of $\mathcal{C}$ is nonempty. (Such a collection $\mathcal{C}$ is said to have the finite intersection property.) Prove that the grand intersection $\bigcap_{C \in \mathcal{C}} C$ is nonempty. [Hint: Consider the collection of open sets $\mathcal{U}=\left\{C^{c}: C \in \mathcal{C}\right.$.]
94. If every collection of closed subsets of $M$ which has the finite intersection property also has a nonempty grand intersection, prove that $M$ is covering compact. [Hint: Given an open covering $\mathcal{U}=\left\{U_{\alpha}\right\}$, consider the collection of closed sets $\mathcal{C}=\left\{U_{\alpha}^{c}\right\}$.]
95. Let $S$ be a subset of a metric space $M$. With respect to the definitions on page 92 prove the following.
(a) The closure of $S$ is the intersection of all closed subsets of $M$ that contain $S$.
(b) The interior of $S$ is the union of all open subsets of $M$ that are contained in $S$.
(c) The boundary of $S$ is a closed set.
(d) Why does (a) imply the closure of $S$ equals $\lim S$ ?
(e) If $S$ is clopen, what is $\partial S$ ?
(f) Give an example of $S \subset \mathbb{R}$ such that $\partial(\partial S) \neq \emptyset$, and infer that "the boundary of the boundary $\partial \circ \partial$ is not always zero."
96. If $A \subset B \subset C, A$ is dense in $B$, and $B$ is dense in $C$ prove that $A$ is dense in $C$.
97. Is the set of dyadic rationals (the denominators are powers of 2) dense in $\mathbb{Q}$ ? In $\mathbb{R}$ ? Does one answer imply the other? (Recall that $A$ is dense in $B$ if $A \subset B$ and $\bar{A} \supset B$.)
98. Show that $S \subset M$ is somewhere dense in $M$ if and only if $\operatorname{int}(\bar{S}) \neq \emptyset$. Equivalently, $S$ is nowhere dense in $M$ if and only if its closure has empty interior.
99. Let $M, N$ be nonempty metric spaces and $P=M \times N$.
(a) If $M, N$ are perfect prove that $P$ is perfect.
(b) If $M, N$ are totally disconnected prove that $P$ is totally disconnected.
(c) What about the converses?
(d) Infer that the Cartesian product of Cantor spaces is a Cantor space. (We already know that the Cartesian product of compacts is compact.)
(e) Why does this imply that $C \times C=\left\{(x, y) \in \mathbb{R}^{2}: x \in C\right.$ and $\left.y \in C\right\}$ is homeomorphic to $C, C$ being the standard Cantor set?
100. Prove that every Cantor piece is a Cantor space. (Recall that $M$ is a Cantor space if it is compact, nonempty, totally disconnected and perfect, and that $A \subset M$ is a Cantor piece if it is nonempty and clopen.)
*101. Let $\Sigma$ be the set of all infinite sequences of zeroes and ones. For example, $(100111000011111 \ldots) \in \Sigma$. Define the metric

$$
d(a, b)=\sum \frac{\left|a_{n}-b_{n}\right|}{2^{n}}
$$

where $a=\left(a_{n}\right)$ and $b=\left(b_{n}\right)$ are points in $\Sigma$.
(a) Prove that $\Sigma$ is compact.
(b) Prove that $\Sigma$ is homeomorphic to the Cantor set.
102. Prove that no Peano curve is one-to-one. (Recall that a Peano curve is a continuous map $f:[0,1] \rightarrow \mathbb{R}^{2}$ whose image has a nonempty interior.)
103. Prove that there is a continuous surjection $\mathbb{R} \rightarrow \mathbb{R}^{2}$. What about $\mathbb{R}^{m}$ ?
104. Find two nonhomeomorphic compact subsets of $\mathbb{R}$ whose complements are homeomorphic.
105. As on page 115 , consider the subsets of $\mathbb{R}$,

$$
A=\{0\} \cup[1,2] \cup\{3\} \quad \text { and } \quad B=\{0\} \cup\{1\} \cup[2,3] .
$$

(a) Why is there no ambient homeomorphism of $\mathbb{R}$ to itself that carries $A$ onto $B$ ?
(b) Thinking of $\mathbb{R}$ as the $x$-axis, is there an ambient homeomorphism of $\mathbb{R}^{2}$ to itself that carries $A$ onto $B$ ?
106. Prove that the completion of a metric space is unique in the following natural sense: A completion of a metric space $M$ is a complete metric $X$ space containing $M$ as a metric subspace such that $M$ is dense in $X$. That is, every point of $X$ is a limit of $M$.
(a) Prove that $M$ is dense in the completion $\widehat{M}$ constructed in the proof of Theorem 80.
(b) If $X$ and $X^{\prime}$ are two completions of $M$ prove that there is an isometry $i: X \rightarrow X^{\prime}$ such that $i(p)=p$ for all $p \in M$.
(c) Prove that $i$ is the unique such isometry.
(d) Infer that $\widehat{M}$ is unique.
107. If $M$ is a metric subspace of a complete metric space $S$ prove that $\bar{M}$ is a completion of $M$.
*108. Consider the identity map id : $C_{\max } \rightarrow C_{\mathrm{int}}$ where $C_{\max }$ is the metric space $C([0,1], \mathbb{R})$ of continuous real-valued functions defined on $[0,1]$, equipped with the max-metric $d_{\max }(f, g)=\max |f(x)-g(x)|$, and $C_{\text {int }}$ is $C([0,1], \mathbb{R})$ equipped with the integral metric,

$$
d_{\mathrm{int}}(f, g)=\int_{0}^{1}|f(x)-g(x)| d x
$$

Show that id is a continuous linear bijection (an isomorphism) but its inverse is not continuous.
*109. A metric on $M$ is an ultrametric if for all $x, y, z \in M$ we have

$$
d(x, z) \leq \max \{d(x, y), d(y, z)\} .
$$

(Intuitively this means that the trip from $x$ to $z$ cannot be broken into shorter legs by making a stopover at some $y$.)
(a) Show that the ultrametric property implies the triangle inequality.
(b) In an ultrametric space show that "all triangles are isosceles."
(c) Show that a metric space with an ultrametric is totally disconnected.
(d) Define a metric on the set $\Sigma$ of strings of zeroes and ones in Exercise 101 as

$$
d_{*}(a, b)= \begin{cases}\frac{1}{2^{n}} & \text { if } n \text { is the smallest index for which } a_{n} \neq b_{n} \\ 0 & \text { if } a=b\end{cases}
$$

Show that $d_{*}$ is an ultrametric and prove that the identity map is a homeomorphism $(\Sigma, d) \rightarrow\left(\Sigma, d_{*}\right)$.
*110. $\mathbb{Q}$ inherits the Euclidean metric from $\mathbb{R}$ but it also carries a very different metric, the $\boldsymbol{p}$-adic metric. Given a prime number $p$ and an integer $n$, the $p$-adic norm of $n$ is

$$
|n|_{p}=\frac{1}{p^{k}}
$$

where $p^{k}$ is the largest power of $p$ that divides $n$. (The norm of 0 is by definition 0 .) The more factors of $p$, the smaller the $p$-norm. Similarly, if $x=a / b$ is a fraction, we factor $x$ as

$$
x=p^{k} \cdot \frac{r}{s}
$$

where $p$ divides neither $r$ nor $s$, and we set

$$
|x|_{p}=\frac{1}{p^{k}}
$$

The $p$-adic metric on $\mathbb{Q}$ is

$$
d_{p}(x, y)=|x-y|_{p}
$$

(a) Prove that $d_{p}$ is a metric with respect to which $\mathbb{Q}$ is perfect - every point is a cluster point.
(b) Prove that $d_{p}$ is an ultrametric.
(c) Let $\mathbb{Q}_{p}$ be the metric space completion of $\mathbb{Q}$ with respect to the metric $d_{p}$, and observe that the extension of $d_{p}$ to $\mathbb{Q}_{p}$ remains an ultrametric. Infer from Exercise 109 that $\mathbb{Q}_{p}$ is totally disconnected.
(d) Prove that $\mathbb{Q}_{p}$ is locally compact, in the sense that every point has small compact neighborhoods.
(e) Infer that $\mathbb{Q}_{p}$ is covered by neighborhoods homeomorphic to the Cantor set. See Gouvêa's book, p-adic Numbers.
111. Let $M=[0,1]$ and let $\mathcal{M}_{1}$ be its division into two intervals $[0,1 / 2]$ and $[1 / 2,1]$. Let $\mathcal{M}_{2}$ be its division into four intervals $[0,1 / 4]$, $[1 / 4,1 / 2]$, $[1 / 2,3 / 4]$, and $[3 / 4,1]$. Continuing these bisections generates natural divisions of $[0,1]$. The pieces are intervals. We label them with words using the letters 0 and 1 as follows: 0 means "left" and 1 means "right," so the four intervals in $\mathcal{N}_{2}$ are labeled as $00,01,10$, and 11 respectively.
(a) Verify that all endpoints of the intervals (except 0 and 1) have two addresses. For instance,

$$
\bigcap_{k}\left[\frac{2^{k-1}-1}{2^{k}}, \frac{1}{2}\right]=\left\{\frac{1}{2}\right\}=\bigcap_{k}\left[\frac{1}{2}, \frac{2^{k-1}+1}{2^{k}}\right]
$$

(b) Verify that the points 0,1 , and all nonendpoints have unique addresses.
*112. Prove that $\# C=\# \mathbb{R}$. [Hint: According to the Schroeder-Bernstein Theorem from Chapter 1 it suffices to find injections $C \rightarrow \mathbb{R}$ and $\mathbb{R} \rightarrow C$. The inclusion $C \subset \mathbb{R}$ is an injection $C \rightarrow \mathbb{R}$. Each $t \in[0,1)$ has a unique base-2 expansion $\tau(t)$ that does not terminate in an infinite string of ones. Replacing each 1 by 2 converts $\tau(t)$ to $\omega(t)$, an infinite address in the symbols 0 and 2 . It does not terminate in an infinite string of twos. Set $h(t)=\sum_{i=1}^{\infty} \omega_{i} / 3^{i}$ and verify that $h:[0,1) \rightarrow C$ is an injection. Since there is an injection $\mathbb{R} \rightarrow[0,1)$, conclude that there is an injection $\mathbb{R} \rightarrow C$, and hence that $\# C=\# \mathbb{R}$.]

Remark The Continuum Hypothesis states that if $S$ is any uncountable subset of $\mathbb{R}$ then $S$ and $\mathbb{R}$ have equal cardinality. The preceding coding shows that $C$ is not only uncountable (as is implied by Theorem 56) but actually has the same cardinality as $\mathbb{R}$. That is, $C$ is not a counterexample to the Continuum Hypothesis. The same is true of all uncountable closed subsets of $\mathbb{R}$. See Exercise 151.
113. Let $M$ be the standard Cantor set $C$. In the notation of Section $8, C^{n}$ is the collection of $2^{n}$ Cantor intervals of length $1 / 3^{n}$ that nest down to $C$ as $n \rightarrow \infty$. Verify that setting $\mathcal{C}_{k}=C \cap C^{k}$ gives divisions of $C$ into disjoint clopen pieces.
*114. (a) Prove directly that there is a continuous surjection of the middle-thirds Cantor set $C$ onto the closed interval $[0,1]$. [Hint: Each $x \in C$ has a base 3 expansion $\left(x_{n}\right)$, all of whose entries are zeroes and twos. (For example, $2 / 3=(2 \overline{0})_{\text {base } 3}$ and $1 / 3=(0 \overline{2})_{\text {base } 3}$. Write $y=\left(y_{n}\right)$ by replacing the twos in $\left(x_{n}\right)$ by ones and interpreting the answer base 2 . Show that the map $x \mapsto y$ works.]
(b) Compare this surjection to the one constructed from the bisection divisions in Exercise 113.
115. Rotate the unit circle $S^{1}$ by a fixed angle $\alpha$, say $R: S^{1} \rightarrow S^{1}$. (In polar coordinates, the transformation $R$ sends $(1, \theta)$ to $(1, \theta+\alpha)$.)
(a) If $\alpha / \pi$ is rational, show that each orbit of $R$ is a finite set.
*(b) If $\alpha / \pi$ is irrational, show that each orbit is infinite and has closure equal to $S^{1}$.
116. A metric space $M$ with metric $d$ can always be remetrized so the metric becomes bounded. Simply define the bounded metric

$$
\rho(p, q)=\frac{d(p, q)}{1+d(p, q)}
$$

(a) Prove that $\rho$ is a metric. Why is it obviously bounded?
(b) Prove that the identity map $M \rightarrow M$ is a homeomorphism from $M$ with the $d$-metric to $M$ with the $\rho$-metric.
(c) Infer that boundedness of $M$ is not a topological property.
(d) Find homeomorphic metric spaces, one bounded and the other not.
117. Fold a piece of paper in half.
(a) Is this a continuous transformation of one rectangle into another?
(b) Is it injective?
(c) Draw an open set in the target rectangle, and find its preimage in the original rectangle. Is it open?
(d) What if the open set meets the crease?

The baker's transformation is a similar mapping. A rectangle of dough is stretched to twice its length and then folded back on itself. Is the transformation continuous? A formula for the baker's transformation in one variable is $f(x)=$ $1-|1-2 x|$. The $\boldsymbol{n}^{\text {th }}$ iterate of $f$ is $f^{n}=f \circ f \circ \ldots \circ f, n$ times. The orbit of a point $x$ is

$$
\left\{x, f(x), f^{2}(x), \ldots, f^{n}(x), \ldots\right\}
$$

[For clearer but more awkward notation one can write $f^{\circ n}$ instead of $f^{n}$. This distinguishes composition $f \circ f$ from multiplication $f \cdot f$.]
(e) If $x$ is rational prove that the orbit of $x$ is a finite set.
(f) If $x$ is irrational what is the orbit?
*118. The implications of compactness are frequently equivalent to it. Prove
(a) If every continuous function $f: M \rightarrow \mathbb{R}$ is bounded then $M$ is compact.
(b) If every continuous bounded function $f: M \rightarrow \mathbb{R}$ achieves a maximum or minimum then $M$ is compact.
(c) If every continuous function $f: M \rightarrow \mathbb{R}$ has compact range $f M$ then $M$ is compact.
(d) If every nested decreasing sequence of nonempty closed subsets of $M$ has nonempty intersection then $M$ is compact.
Together with Theorems 63 and 65, (a)-(d) give seven equivalent definitions of compactness. [Hint: Reason contrapositively. If $M$ is not compact then it contains a sequence $\left(p_{n}\right)$ that has no convergent subsequence. It is fair to assume that the points $p_{n}$ are distinct. Find radii $r_{n}>0$ such that the neighborhoods $M_{r_{n}}\left(p_{n}\right)$ are disjoint and no sequence $q_{n} \in M_{r_{n}}\left(p_{n}\right)$ has a convergent subsequence. Using the metric define a function $f_{n}: M_{r_{n}}\left(p_{n}\right) \rightarrow \mathbb{R}$ with a spike at $p_{n}$, such as

$$
f_{n}(x)=\frac{r_{n}-d\left(x, p_{n}\right)}{a_{n}+d\left(x, p_{n}\right)}
$$

where $a_{n}>0$. Set $f(x)=f_{n}(x)$ if $x \in M_{r_{n}}\left(p_{n}\right)$, and $f(x)=0$ if $x$ belongs to no $M_{r_{n}}\left(p_{n}\right)$. Show that $f$ is continuous. With the right choice of $a_{n}$ show that $f$ is unbounded. With a different choice of $a_{n}$, it is bounded but achieves no maximum, and so on.]
119. Let $M$ be a metric space of diameter $\leq 2$. The cone for $M$ is the set

$$
C=C(M)=\left\{p_{0}\right\} \cup M \times(0,1]
$$

with the cone metric

$$
\begin{aligned}
\rho((p, s),(q, t)) & =|s-t|+\min \{s, t\} d(p, q) \\
\rho\left((p, s), p_{0}\right) & =s \\
\rho\left(p_{0}, p_{0}\right) & =0 .
\end{aligned}
$$

The point $p_{0}$ is the vertex of the cone. Prove that $\rho$ is a metric on $C$. [If $M$ is the unit circle, think of it in the plane $z=1$ in $\mathbb{R}^{3}$ centered at the point $(0,0,1)$. Its cone is the 45 -degree cone with vertex the origin.]
120. Recall that if for each embedding of $M, h: M \rightarrow N, h M$ is closed in $N$ then $M$ is said to be absolutely closed. If each $h M$ is bounded then $M$ is absolutely bounded. Theorem 41 implies that compact sets are absolutely closed and absolutely bounded. Prove:
(a) If $M$ is absolutely bounded then $M$ is compact.
*(b) If $M$ is absolutely closed then $M$ is compact.
Thus these are two more conditions equivalent to compactness. [Hint: From Exercise 118(a), if $M$ is noncompact there is a continuous function $f: M \rightarrow \mathbb{R}$ that is unbounded. For Exercise 120(a), show that $F(x)=(x, f(x))$ embeds $M$ onto a nonbounded subset of $M \times \mathbb{R}$. For $120(\mathrm{~b})$, justify the additional assumption that the metric on $M$ is bounded by 2. Then use Exercise 118(b) to show that if $M$ is noncompact then there is a continuous function $g: M \rightarrow(0,1]$ such that for some nonclustering sequence $\left(p_{n}\right)$, we have $g\left(p_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$. Finally, show that $G(x)=(x, g x)$ embeds $M$ onto a nonclosed subset $S$ of the
cone $C(M)$ discussed in Exercise 119. $S$ will be nonclosed because it limits at $p_{0}$ but does not contain it.]
121. (a) Prove that every function defined on a discrete metric space is uniformly continuous.
(b) Infer that it is false to assert that if every continuous function $f: M \rightarrow \mathbb{R}$ is uniformly continuous then $M$ is compact.
(c) Prove, however, that if $M$ is a metric subspace of a compact metric space $K$ and every continuous function $f: M \rightarrow \mathbb{R}$ is uniformly continuous then $M$ is compact.
122. Recall that $p$ is a cluster point of $S$ if each $M_{r} p$ contains infinitely many points of $S$. The set of cluster points of $S$ is denoted as $S^{\prime}$. Prove:
(a) If $S \subset T$ then $S^{\prime} \subset T^{\prime}$.
(b) $(S \cup T)^{\prime}=S^{\prime} \cup T^{\prime}$.
(c) $S^{\prime}=(\bar{S})^{\prime}$.
(d) $S^{\prime}$ is closed in $M$; that is, $S^{\prime \prime} \subset S^{\prime}$ where $S^{\prime \prime}=\left(S^{\prime}\right)^{\prime}$.
(e) Calculate $\mathbb{N}^{\prime}, \mathbb{Q}^{\prime}, \mathbb{R}^{\prime},(\mathbb{R} \backslash \mathbb{Q})^{\prime}$, and $\mathbb{Q}^{\prime \prime}$.
(f) Let $T$ be the set of points $\{1 / n: n \in \mathbb{N}\}$. Calculate $T^{\prime}$ and $T^{\prime \prime}$.
(g) Give an example showing that $S^{\prime \prime}$ can be a proper subset of $S^{\prime}$.
123. Recall that $p$ is a condensation point of $S$ if each $M_{r} p$ contains uncountably many points of $S$. The set of condensation points of $S$ is denoted as $S^{*}$. Prove:
(a) If $S \subset T$ then $S^{*} \subset T^{*}$.
(b) $(S \cup T)^{*}=S^{*} \cup T^{*}$.
(c) $S^{*} \subset \bar{S}^{*}$ where $\bar{S}^{*}=(\bar{S})^{*}$
(d) $S^{*}$ is closed in $M$; that is, $S^{* \prime} \subset S^{*}$ where $S^{* \prime}=\left(S^{*}\right)^{\prime}$.
(e) $S^{* *} \subset S^{*}$ where $S^{* *}=\left(S^{*}\right)^{*}$.
(f) Calculate $\mathbb{N}^{*}, \mathbb{Q}^{*}, \mathbb{R}^{*}$, and $(\mathbb{R} \backslash \mathbb{Q})^{*}$.
(g) Give an example showing that $S^{*}$ can be a proper subset of $(\bar{S})^{*}$. Thus, (c) is not in general an equality.
${ }^{* *}(\mathrm{~h})$ Give an example that $S^{* *}$ can be a proper subset of $S^{*}$. Thus, (e) is not in general an equality. [Hint: Consider the set $M$ of all functions $f:[a, b] \rightarrow[0,1]$, continuous or not, and let the metric on $M$ be the sup metric, $d(f, g)=\sup \{|f(x)-g(x)|: x \in[a, b]\}$. Consider the set $S$ of all " $\delta$-functions with rational values."]
${ }^{* *}$ (i) Give examples that show in general that $S^{*}$ neither contains nor is contained in $S^{\prime *}$ where $S^{\prime *}=\left(S^{\prime}\right)^{*}$. [Hint: $\delta$-functions with values $1 / n, n \in \mathbb{N}$.]
124. Recall that $p$ is an interior point of $S \subset M$ if some $M_{r} p$ is contained in $S$. The set of interior points of $S$ is the interior of $S$ and is denoted int $S$. For all subsets $S, T$ of the metric space $M$ prove:
(a) $\operatorname{int} S=S \backslash \partial S$.
(b) int $S=\left(\overline{S^{c}}\right)^{c}$.
(c) $\operatorname{int}(\operatorname{int} S)=\operatorname{int} S$.
(d) $\operatorname{int}(S \cap T)=\operatorname{int}(S \cap \operatorname{int} T$.
(e) What are the dual equations for the closure?
(f) Prove that $\operatorname{int}(S \cup T) \supset \operatorname{int} S \cup \operatorname{int} T$. Show by example that the inclusion can be strict, i.e., not an equality.
125. A point $p$ is a boundary point of a set $S \subset M$ if every neighborhood $M_{r} p$ contains points of both $S$ and $S^{c}$. The boundary of $S$ is denoted $\partial S$. For all subsets $S, T$ of a metric space $M$ prove:
(a) $S$ is clopen if and only if $\partial S=\emptyset$.
(b) $\partial S=\partial S^{c}$.
(c) $\partial \partial S \subset \partial S$.
(d) $\partial \partial \partial S=\partial \partial S$.
(e) $\partial(S \cup T) \subset \partial S \cup \partial T$.
(f) Give an example in which (c) is a strict inclusion, $\partial \partial S \neq \partial S$.
(g) What about (e)?
*126. Suppose that $E$ is an uncountable subset of $\mathbb{R}$. Prove that there exists a point $p \in \mathbb{R}$ at which $E$ condenses. [Hint: Use decimal expansions. Why must there be an interval $[n, n+1)$ containing uncountably many points of $E$ ? Why must it contain a decimal subinterval with the same property? (A decimal subinterval $[a, b)$ has endpoints $a=n+k / 10, b=n+(k+1) / 10$ for some digit $k, 0 \leq k \leq 9$.) Do you see lurking the decimal expansion of a condensation point?] Generalize to $\mathbb{R}^{2}$ and to $\mathbb{R}^{m}$.
127. The metric space $M$ is separable if it contains a countable dense subset. [Note the confusion of language: "Separable" has nothing to do with "separation."]
(a) Prove that $\mathbb{R}^{m}$ is separable.
(b) Prove that every compact metric space is separable.
128. *(a) Prove that every metric subspace of a separable metric space is separable, and deduce that every metric subspace of $\mathbb{R}^{m}$ or of a compact metric space is separable.
(b) Is the property of being separable topological?
(c) Is the continuous image of a separable metric space separable?
129. Think up a nonseparable metric space.
130. Let $\mathcal{B}$ denote the collection of all $\epsilon$-neighborhoods in $\mathbb{R}^{m}$ whose radius $\epsilon$ is rational and whose center has all coordinates rational.
(a) Prove that $\mathcal{B}$ is countable.
(b) Prove that every open subset of $\mathbb{R}^{m}$ can be expressed as the countable union of members of $\mathcal{B}$.
(The union need not be disjoint, but it is at most a countable union because there are only countably many members of $\mathcal{B}$. A collection such as $\mathcal{B}$ is called a countable base for the topology of $\mathbb{R}^{m}$.)
131. (a) Prove that every separable metric space has a countable base for its topology, and conversely that every metric space with a countable base for its topology is separable.
(b) Infer that every compact metric space has a countable base for its topology.
*132. Referring to Exercise 123, assume now that $M$ is separable, $S \subset M$, and, as before $S^{\prime}$ is the set of cluster points of $S$ while $S^{*}$ is the set of condensation points of $S$. Prove:
(a) $S^{*} \subset\left(S^{\prime}\right)^{*}=(\bar{S})^{*}$.
(b) $S^{* *}=S^{* \prime}=S^{*}$.
(c) Why is (a) not in general an equality?
[Hints: For (a) write $S \subset\left(S \backslash S^{\prime}\right) \cup S^{\prime}$ and $\bar{S}=\left(S \backslash S^{\prime}\right) \cup S^{\prime}$, show that $\left(S \backslash S^{\prime}\right)^{*}=\emptyset$, and use Exercise 123(a). For (b), Exercise 123(d) implies that $S^{* *} \subset S^{* \prime} \subset S^{*}$. To prove that $S^{*} \subset S^{* *}$, write $S \subset\left(S \backslash S^{*}\right) \cup S^{*}$ and show that $\left.\left(S \backslash S^{*}\right)^{*}=\emptyset.\right]$
*133. Prove that
(a) An uncountable subset of $\mathbb{R}$ clusters at some point of $\mathbb{R}$.
(b) An uncountable subset of $\mathbb{R}$ clusters at some point of itself.
(c) An uncountable subset of $\mathbb{R}$ condenses at uncountably many points of itself.
(d) What about $\mathbb{R}^{m}$ instead of $\mathbb{R}$ ?
(e) What about any compact metric space?
(f) What about any separable metric space?
[Hint: Review Exercise 126.]
*134. Prove that $\widehat{\mathbb{Q}}$, the Cauchy sequences in $\mathbb{Q}$ modulo the equivalence relation of being co-Cauchy, is a field with respect to the natural arithmetic operations defined on page 122 , and that $\mathbb{Q}$ is naturally a subfield of $\widehat{\mathbb{Q}}$.
135. Prove that the order on $\widehat{\mathbb{Q}}$ defined on page 122 is a bona fide order which agrees with the standard order on $\mathbb{Q}$.
*136. Let $M$ be the square $[0,1]^{2}$, and let $a a, b a, b b, a b$ label its four quadrants - upper right, upper left, lower left, and lower right.
(a) Define nested bisections of the square using this pattern repeatedly, and let $\tau_{k}$ be a curve composed of line segments that visit the $k^{\text {th }}$-order quadrants systematically. Let $\tau=\lim _{k} \tau_{k}$ be the resulting Peano curve à la the Cantor Surjection Theorem.
(b) Compare $\tau$ to the Peano curve $f: I \rightarrow I^{2}$ directly constructed on pages 271-274 of the second edition of Munkres' book Topology.
*137. Let $P$ be a closed perfect subset of a separable complete metric space $M$. Prove that each point of $P$ is a condensation point of $P$. In symbols, $P=P^{\prime} \Rightarrow$ $P=P^{*}$.
${ }^{* *}$ 138. Given a Cantor space $M \subset \mathbb{R}^{2}$, given a line segment $[p, q] \subset \mathbb{R}^{2}$ with $p, q \notin M$,
and given an $\epsilon>0$, prove that there exists a path $A$ in the $\epsilon$-neighborhood of $[p, q]$ that joins $p$ to $q$ and is disjoint from $M$. [Hint: Think of $A$ as a bisector of $M$. From this bisection fact a dyadic disc partition of $M$ can be constructed, which leads to the proof that $M$ is tame.]
139. To prove that Antoine's Necklace $A$ is a Cantor set, you need to show that $A$ is compact, perfect, nonempty, and totally disconnected.
(a) Do so. [Hint: What is the diameter of any connected component of $A^{n}$, and what does that imply about $A$ ?]
**(b) If, in the Antoine construction two linked solid tori are placed very cleverly inside each larger solid torus, show that the intersection $A=\bigcap A^{n}$ is a Cantor set.
*140. Consider the Hilbert cube

$$
H=\left\{\left(x_{1}, x_{2}, \ldots\right) \in[0,1]^{\infty}: \text { for each } n \in \mathbb{N} \text { we have }\left|x_{n}\right| \leq 1 / 2^{n}\right\}
$$

Prove that $H$ is compact with respect to the metric

$$
d(x, y))=\sup _{n}\left|x_{n}-y_{n}\right|
$$

where $x=\left(x_{n}\right), y=\left(y_{n}\right)$. [Hint: Sequences of sequences.]
Remark Although compact, $H$ is infinite-dimensional and is homeomorphic to no subset of $\mathbb{R}^{m}$.
141. Prove that the Hilbert cube is perfect and homeomorphic to its Cartesian square, $H \cong H \times H$.
***142. Assume that $M$ is compact, nonempty, perfect, and homeomorphic to its Cartesian square, $M \cong M \times M$. Must $M$ be homeomorphic to the Cantor set, the Hilbert cube, or some combination of them?
143. A Peano space is a metric space $M$ that is the continuous image of the unit interval: There is a continuous surjection $\tau:[0,1] \rightarrow M$. Theorem 72 states the amazing fact that the 2-disc is a Peano space. Prove that every Peano space is
(a) compact,
(b) nonempty,
(c) path-connected,
*(d) and locally path-connected, in the sense that for each $p \in M$ and each neighborhood $U$ of $p$ there is a smaller neighborhood $V$ of $p$ such that any two points of $V$ can be joined by a path in $U$.
*144. The converse to Exercise 143 is the Hahn-Mazurkiewicz Theorem. Assume that a metric space $M$ is a compact, nonempty, path-connected, and locally path-connected. Use the Cantor Surjection Theorem 70 to show that $M$ is a Peano space. [The key is to make uniformly short paths to fill in the gaps of $[0,1] \backslash C$.
145. One of the famous theorems in plane topology is the Jordan Curve Theorem. A Jordan curve $J$ is a homeomorph of the unit circle in the plane. (Equivalently it is $f([a, b])$ where $f:[a, b] \rightarrow \mathbb{R}^{2}$ is continuous, $f(a)=f(b)$, and for no other pair of distinct $s, t \in[a, b]$ does $f(s)$ equal $f(t)$. It is also called a simple closed curve.) The Jordan Curve Theorem asserts that $\mathbb{R}^{2} \backslash J$ consists of two disjoint, connected open sets, its inside and its outside, and every path between them must meet $J$. Prove the Jordan Curve Theorem for the circle, the square, the triangle, and - if you have courage - every simple closed polygon.
146. The utility problem gives three houses $1,2,3$ in the plane and the three utilities, Gas, Water, and Electricity. You are supposed to connect each house to the three utilities without crossing utility lines. (The houses and utilities are disjoint.)
(a) Use the Jordan curve theorem to show that there is no solution to the utility problem in the plane.
*(b) Show also that the utility problem cannot be solved on the 2 -sphere $S^{2}$.
*(c) Show that the utility problem can be solved on the surface of the torus.
*(d) What about the surface of the Klein bottle?
***(e) Given utilities $U_{1}, \ldots, U_{m}$ and houses $H_{1}, \ldots, H_{n}$ located on a surface with $g$ handles, find necessary and sufficient conditions on $m, n, g$ so that the utility problem can be solved.
147. Let $M$ be a metric space and let $\mathcal{K}$ denote the class of nonempty compact subsets of $M$. The $r$-neighborhood of $A \in \mathscr{K}$ is

$$
M_{r} A=\{x \in M: \exists a \in A \text { and } d(x, a)<r\}=\mathbf{U}_{a \in A} M_{r} a
$$

For $A, B \in \mathcal{K}$ define

$$
D(A, B)=\inf \left\{r>0: A \subset M_{r} B \text { and } B \subset M_{r} A\right\}
$$

(a) Show that $D$ is a metric on $\mathcal{K}$. (It is called the Hausdorff metric and $\mathcal{K}$ is called the hyperspace of $M$.)
(b) Denote by $\mathcal{F}$ the collection of finite nonempty subsets of $M$ and prove that $\mathcal{F}$ is dense in $\mathcal{K}$. That is, given $A \in \mathcal{K}$ and given $\epsilon>0$ show there exists $F \in \mathcal{F}$ such that $D(A, F)<\epsilon$.
*(c) If $M$ is compact prove that $\mathcal{K}$ is compact.
(d) If $M$ is connected prove that $\mathcal{K}$ is connected.
**(e) If $M$ is path-connected is $\mathcal{K}$ path-connected?
(f) Do homeomorphic metric spaces have homeomorphic hyperspaces?

Remark The converse to (f), $\mathcal{K}(M) \cong \mathcal{K}(N) \Rightarrow M \cong N$ is false. The hyperspace of every Peano space is the Hilbert cube. This is a difficult result but a good place to begin reading about hyperspaces is Sam Nadler's book Continuum Theory.
${ }^{* *}$ 148. Start with a set $S \subset \mathbb{R}$ and successively take its closure, the complement of its closure, the closure of that, and so on: $S, \operatorname{cl}(S),(\operatorname{cl}(S))^{c}, \ldots$ Do the same to $S^{c}$. In total, how many distinct subsets of $\mathbb{R}$ can be produced this way? In particular decide whether each chain $S, \operatorname{cl}(S), \ldots$ consists of only finitely many sets. For example, if $S=\mathbb{Q}$ then we get $\mathbb{Q}, \mathbb{R}, \emptyset, \emptyset, \mathbb{R}, \mathbb{R}, \ldots$ and $\mathbb{Q}^{c}, \mathbb{R}, \emptyset, \emptyset, \mathbb{R}, \mathbb{R}, \ldots$ for a total of four sets.
${ }^{* *}$ 149. Consider the letter T.
(a) Prove that there is no way to place uncountably many copies of the letter T disjointly in the plane. [Hint: First prove this when the unit square replaces the plane.]
(b) Prove that there is no way to place uncountably many homeomorphic copies of the letter T disjointly in the plane.
(c) For which other letters of the alphabet is this true?
(d) Let $U$ be a set in $\mathbb{R}^{3}$ formed like an umbrella: It is a disc with a perpendicular segment attached to its center. Prove that uncountably many copies of $U$ cannot be placed disjointly in $\mathbb{R}^{3}$.
(e) What if the perpendicular segment is attached to the boundary of the disc?
${ }^{* *} 150$. Let $M$ be a complete, separable metric space such as $\mathbb{R}^{m}$. Prove the Cupcake Theorem: Each closed set $K \subset M$ can be expressed uniquely as the disjoint union of a countable set and a perfect closed set,

$$
C \sqcup P=K
$$

${ }^{* *} 151$. Let $M$ be an uncountable compact metric space.
(a) Prove that $M$ contains a homeomorphic copy of the Cantor set. [Hint: Imitate the construction of the standard Cantor set C.]
(b) Infer that Cantor sets are ubiquitous. There is a continuous surjection $\sigma: C \rightarrow M$ and there is a continuous injection $i: C \rightarrow M$.
(c) Infer that every uncountable closed set $S \subset \mathbb{R}$ has $\# S=\# \mathbb{R}$, and hence that the Continuum Hypothesis is valid for closed sets in $\mathbb{R}$. [Hint: Cupcake and Exercise 112.]
(d) Is the same true if $M$ is separable, uncountable, and complete?
${ }^{* *} 152$. Write jingles at least as good as the following. Pay attention to the meter as well as the rhyme.

When a set in the plane is closed and bounded, you can always draw a curve around it.

If a clopen set can be detected, Your metric space is disconnected.

David Owens

A coffee cup feeling quite dazed, said to a donut, amazed, an open surjective continuous injection, You'd be plastic and I'd be glazed.

Norah Esty

'Tis a most indisputable fact
If you want to make something compact
Make it bounded and closed
For you're totally hosed
If either condition you lack.
Lest the reader infer an untruth
(Which I think would be highly uncouth)
I must hasten to add
There are sets to be had
Where the converse is false, fo'sooth.

Karla Westfahl

For ev'ry $a$ and $b$ in $S$
if there exists a path that's straight
from $a$ to $b$ and it's inside
then " $S$ must be convex," we state.
Alex Wang

## Prelim Problems ${ }^{\dagger}$

1. Suppose that $f: \mathbb{R}^{m} \rightarrow \mathbb{R}$ satisfies two conditions:
(i) For each compact set $K, f(K)$ is compact.
(ii) For every nested decreasing sequence of compacts $\left(K_{n}\right)$,

$$
f\left(\mathbf{\Omega} K_{n}\right)=\mathbf{\bigcap} f\left(K_{n}\right) .
$$

Prove that $f$ is continuous.
2. Let $X \subset \mathbb{R}^{m}$ be compact and $f: X \rightarrow \mathbb{R}$ be continuous. Given $\epsilon>0$, show that there is a constant $M$ such that for all $x, y \in X$ we have $|f(x)-f(y)| \leq$ $M|x-y|+\epsilon$.
3. Consider $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$. Assume that for each fixed $x_{0}, y \mapsto f\left(x_{0}, y\right)$ is continuous and for each fixed $y_{0}, x \mapsto f\left(x, y_{0}\right)$ is continuous. Find such an $f$ that is not continuous.
4. Let $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ satisfy the following properties. For each fixed $x_{0} \in \mathbb{R}$ the function $y \mapsto f\left(x_{0}, y\right)$ is continuous and for each fixed $y_{0} \in \mathbb{R}$ the function $x \mapsto f\left(x, y_{0}\right)$ is continuous. Also assume that if $K$ is any compact subset of $\mathbb{R}^{2}$ then $f(K)$ is compact. Prove that $f$ is continuous.
5. Let $f(x, y)$ be a continuous real-valued function defined on the unit square $[0,1] \times[0,1]$. Prove that

$$
g(x)=\max \{f(x, y): y \in[0,1]\}
$$

is continuous.
6. Let $\left\{U_{k}\right\}$ be a cover of $\mathbb{R}^{m}$ by open sets. Prove that there is a cover $\left\{V_{k}\right\}$ of $\mathbb{R}^{m}$ by open sets $V_{k}$ such that $V_{k} \subset U_{k}$ and each compact subset of $\mathbb{R}^{m}$ is disjoint from all but finitely many of the $V_{k}$.
7. A function $f:[0,1] \rightarrow \mathbb{R}$ is said to be upper semicontinuous if given $x \in[0,1]$ and $\epsilon>0$ there exists a $\delta>0$ such that $|y-x|<\delta$ implies that $f(y)<f(x)+\epsilon$. Prove that an upper semicontinuous function on $[0,1]$ is bounded above and attains its maximum value at some point $p \in[0,1]$.
8. Prove that a continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ which sends open sets to open sets must be monotonic.
9. Show that $[0,1]$ cannot be written as a countably infinite union of disjoint closed subintervals.
10. A connected component of a metric space $M$ is a maximal connected subset of $M$. Give an example of $M \subset \mathbb{R}$ having uncountably many connected components. Can such a subset be open? Closed? Does your answer change if $\mathbb{R}^{2}$ replaces $\mathbb{R}$ ?

[^9]11. Let $U \subset \mathbb{R}^{m}$ be an open set. Suppose that the map $h: U \rightarrow \mathbb{R}^{m}$ is a homeomorphism from $U$ onto $\mathbb{R}^{m}$ which is uniformly continuous. Prove that $U=\mathbb{R}^{m}$.
12. Let $X$ be a nonempty connected set of real numbers. If every element of $X$ is rational prove that $X$ has only one element.
13. Let $A \subset \mathbb{R}^{m}$ be compact, $x \in A$. Let $\left(x_{n}\right)$ be a sequence in $A$ such that every convergent subsequence of $\left(x_{n}\right)$ converges to $x$.
(a) Prove that the sequence $\left(x_{n}\right)$ converges.
(b) Give an example to show if $A$ is not compact, the result in (a) is not necessarily true.
14. Assume that $f: \mathbb{R} \rightarrow \mathbb{R}$ is uniformly continuous. Prove that there are constants $A, B$ such that $|f(x)| \leq A+B|x|$ for all $x \in \mathbb{R}$.
15. Let $h:[0,1) \rightarrow \mathbb{R}$ be a uniformly continuous function where $[0,1)$ is the halfopen interval. Prove that there is a unique continuous map $g:[0,1] \rightarrow \mathbb{R}$ such that $g(x)=h(x)$ for all $x \in[0,1)$.


[^0]:    ${ }^{\dagger}$ This is a rare case in mathematics in which spelling is important. Homeomorphism $\neq$ homomorphism.

[^1]:    ${ }^{\dagger}$ A limit of $S$ is also sometimes called a limit point of $S$. Take care though: Some mathematicians require that a limit point of $S$ be the limit of a sequence of distinct points of $S$. They would say that a finite set has no limit points. We will not adopt their point of view. Another word used in this context, especially by the French, is "adherence." A point $p$ adheres to the set $S$ if and only if $p$ is a limit of $S$. In more general circumstances, limits are defined using "nets" instead of sequences. They are like "uncountable sequences." You can read more about nets in graduate-level topology books such as Topology by James Munkres.

[^2]:    ${ }^{\dagger}$ Note how similarly algebraists use the word "closed." A field (or group or ring, etc.) is closed under its arithmetic operations: Sums, differences, products, and quotients of elements in the field still lie in the field. In our case it is limits. Limits of sequences in $S$ must lie in $S$.

[^3]:    ${ }^{\dagger}$ Any collection $\mathcal{T}$ of subsets of a set $X$ that satisfies these three properties is called a topology on $X$, and $X$ is called a topological space. Topological spaces are more general than metric spaces: There exist topologies that do not arise from a metric. Think of them as pathological. The question of which topologies can be generated by a metric and which cannot is discussed in Topology by Munkres. See also Exercise 30.
    ${ }^{\ddagger}$ The $\alpha$ in the notation $U_{\alpha}$ "indexes" the sets. If $\alpha=1,2, \ldots$ then the collection is countable, but we are just as happy to let $\alpha$ range through uncountable index sets.

[^4]:    ${ }^{\dagger}$ The preimage of $V$ is also called the inverse image of $V$ and is denoted by $f^{-1}(V)$. Unless $f$ is a bijection, this notation leads to confusion. There may be no map $f^{-1}$ and yet expressions like $V \supset f\left(f^{-1}(V)\right)$ are written that mix maps and nonmaps. By the way, if $f$ sends no point of $M$ into $V$ then $f^{\text {pre }}(V)$ is the empty set.

[^5]:    ${ }^{\dagger}$ I have asked variants of the following True or False question on every analysis exam I've given: "Every closed and bounded subset of a complete metric space is compact." You would be surprised at how many students answer "True."

[^6]:    ${ }^{\dagger}$ Cluster points are also called accumulation points. As mentioned above, they are also sometimes called limit points, a usage that conflicts with the limit idea. A finite set $S$ has no cluster points, but of course, each of its points $p$ is a limit of $S$ since the constant sequence $(p, p, p, \ldots)$ converges to $p$.

[^7]:    ${ }^{\dagger}$ You will frequently find it said that an open covering of $A$ has a finite subcovering. "Has" means "reduces to."

[^8]:    ${ }^{\dagger}$ This is nicely is expressed by Pierre Teilhard de Chardin, "Tout ce qui monte converge," in a different context.

[^9]:    ${ }^{\dagger}$ These are questions taken from the exam given to first-year math graduate students at U.C. Berkeley.

