104 HW 10

33.4 Consider $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$

This not integrable, but $|f|$ is 1 throughout $[0,1]$ and is integrable

33.7

(a) $U(f^2, P) - L(f^2, P)$

$$= \sum_{i=1}^{N} \Delta x_i \left(\sup_{[x_{i-1}, x_i]} f(x)^2 - \inf_{[x_{i-1}, x_i]} f(x)^2 \right)$$

$$= \sum_{i=1}^{N} \Delta x_i \left(\left(\sup_{[x_{i-1}, x_i]} f(x)\right)^2 - \left(\inf_{[x_{i-1}, x_i]} f(x)\right)^2 \right)$$

$$= \sum_{i=1}^{N} \Delta x_i \left(\sup_{[x_{i-1}, x_i]} f(x) + \inf_{[x_{i-1}, x_i]} f(x) \right) \left(\sup_{[x_{i-1}, x_i]} f(x) - \inf_{[x_{i-1}, x_i]} f(x) \right)$$

Since $\sup_{[x_{i-1}, x_i]} f(x) \leq B$ and $\inf_{[x_{i-1}, x_i]} f(x) \leq B$

$$\leq \sum_{i=1}^{N} \Delta x_i (2B) \left(\sup_{[x_{i-1}, x_i]} f(x) - \inf_{[x_{i-1}, x_i]} f(x) \right)$$

$$= 2B \left[U(f,P) - L(f,P) \right]$$

(b)

If f is integrable, there exists a partition of P of $[a,b]$ such that $|U(f,P) - L(f,P)| < \frac{\varepsilon}{2B}$ for any $\varepsilon > 0$

$U(f^2,P) - L(f^2,P) \leq 2B|U(f,P) - L(f,P)| < \varepsilon$ as desired

$\Rightarrow f^2$ is integrable

33.13 Assume for contradiction that there is no x in (a,b) s.t. $f(x) = g(x)$. Then $f(x) < g(x)$ for all x in (a,b) or $f(x) > g(x)$ $\forall x \in (a,b)$

In either case $\int_a^b f(x) > \int_a^b g(x)$ or $\int_a^b g(x) > \int_a^b f(x)$ which is a contradiction

So $\exists$ x in (a,b) s.t. $f(x) = g(x)$

35-4

(a) $\int_0^{\pi/2} x \, dF(x) \qquad F(x) = \sin x$

By 35.19 Thm,

$$\int_0^{\pi/2} x \, dF(x) + \int_0^{\pi/2} F(x)\, dx = \frac{\pi}{2} \cdot \sin\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \int_0^{\pi/2} x \, dF(x) = \frac{\pi}{2} - (-\cos x)\Big|_0^{\pi/2} = \boxed{\frac{\pi}{2} - 1}$$

(b) $$\int_{-\pi/2}^{\pi/2} x \, dF(x) + \int_{-\pi/2}^{\pi/2} F(x)\, dx = \frac{\pi}{2} \sin\left(\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) = 0$$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} x \, dF(x) = -(-\cos x)\Big|_{-\pi/2}^{\pi/2} = \boxed{0}$$

35-9 (a)

we know $\left(\inf_{[a,b]} f\right)(F(b) - F(a)) \leq \int_a^b f \, dF \leq \left(\sup_{[a,b]} f\right)(F(b) - F(a))$

So since ~~there are~~ $f(x)$ is continuous, there is some x such that

$$f(x)(F(b) - F(a)) = \int_a^b f \, dF$$

by IVT