104 HW 8

① I claim $f_n \to \frac{1}{2}$ uniformly.

$$\frac{n-1}{2n+1} \le f_n(x) \le \frac{n+1}{2n-1}$$

For any $\varepsilon > 0$, we want $|f_n(x) - f(x)| < \varepsilon$

$$\Leftrightarrow \left|f_n(x) - \tfrac{1}{2}\right| < \varepsilon$$

so we want

$$\frac{n+1}{2n-1} - \frac{1}{2} < \varepsilon$$

and

$$\frac{1}{2} - \frac{n-1}{2n+1} < \varepsilon$$

$$\frac{3/2}{2n-1} < \varepsilon \Rightarrow \boxed{n > \frac{\frac{3}{2}+\varepsilon}{2\varepsilon}}$$

and

$$\frac{3/2}{2n+1} < \varepsilon$$

So consider $N = \frac{3}{4\varepsilon} + \frac{1}{2}$

then $n > N \Rightarrow \left|f_n(x) - \frac{1}{2}\right| \le \left|\frac{n+1}{2n-1} - \frac{1}{2}\right|$

$$= \left|\frac{3/2}{2n-1}\right| < \varepsilon$$

So f_n converges uniformly on $\mathbb{R}$.

$$f_n = \sum_{i=1}^{n} a_i x^n$$

② We know $\left|\sum_{n=1}^{\infty} a_n x^n\right| \le |a_1 x^1| + |a_2 x^2| + \dots \le |a_1| + |a_2| + \dots < \infty$ so $f(x)$ is bounded.

~~also since f_n is a sum of polynomials, it is continuous~~

Also, since the partial sum of polynomials is a finite sum of polynomials, it is continuous.

Now it remains to show f_n uniformly converges to f.

Consider $\varepsilon > 0$ $\quad |f_n - f| = \left|\sum_{i=n+1}^{\infty} a_i x^i\right|$

~~this is true since we can just choose N index of largest absolute value.~~

let $z = \max\{|a_{n+1}|, |a_{n+2}|, \dots\}$ since Σ is bounded

then $\left|\sum_{i=n+1}^{\infty} a_i x^i\right| < \left|\sum_{i=n+1} z x^i\right| = z \cdot \frac{|x|^{n+1}}{1-|x|}$

So we can choose our n accordingly

$= \frac{\varepsilon}{z}$ not in terms of x

so that inequality is satisfied

So, by Thm, f is continuous.

We can use the Same Thm Since $\left|\sum_{n=1}^{\infty} n^{-2} x^n\right| \le \sum_{n=1}^{\infty} |n^{-2} x^n| \le \sum_{n=1}^{\infty} |n^{-2}| = \frac{\pi^2}{6}$

So since the sum is bounded, we can use the same Thm to say $\sum_{n=1}^{\infty} n^{-2} x^n$ is continuous on $[-1, 1]$

③ We first show that for $0<a<1$, we have uniform convergence on $[-a, a]$

~~Weierstrass~~

Let $M_n = a^n$ $\quad \sum_{n=1}^{\infty} M_n$ converges and $|x^n| \le |a^n|$

so by Weierstrass M-test, $\sum_{n=1}^{\infty} x^n$ converges uniformly on $[-a, a]$

$|x| \le a$

$\Rightarrow$ $f(x)$ converges on $(-1, 1)$

However consider

$$|f_n(x) - f(x)| = \left|\sum_{n=n+1}^{\infty} x^n\right| = \frac{x^{n+1}}{1-x}$$

which can get arbitrarily large as $|x| \to 1$

So the ~~~~ N must depend on x to define the convergence

$\Rightarrow$ it is not uniform.