

① I claim  $f_n \rightarrow \frac{1}{2}$  uniformly.

$$\frac{n-1}{2n+1} \leq f_n(x) \leq \frac{n+1}{2n-1}$$

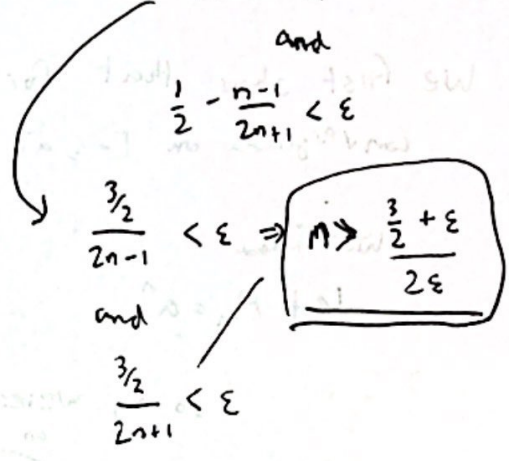
For any  $\epsilon > 0$ , we want  $|f_n(x) - \frac{1}{2}| < \epsilon$   
 $\Leftrightarrow |f_n(x) - \frac{1}{2}| < \epsilon$

so we want  
 $\frac{n+1}{2n-1} - \frac{1}{2} < \epsilon$   
 and  
 $\frac{1}{2} - \frac{n-1}{2n+1} < \epsilon$

So consider  $N \approx \frac{3}{4\epsilon} + \frac{1}{2}$

then  $n > N \Rightarrow |f_n(x) - \frac{1}{2}| \leq \left| \frac{n+1}{2n-1} - \frac{1}{2} \right|$

$$= \left| \frac{3/2}{2n-1} \right| < \epsilon$$



So  $f_n$  converges uniformly on  $\mathbb{R}$ .

$$f_n = \sum_{i=1}^n a_i x^i$$

② We know  $|\sum_{n=1}^{\infty} a_n x^n| \leq |a_1 x^1| + |a_2 x^2| + \dots \leq |a_1| + |a_2| + \dots < \infty$  so  $f_n(x)$  is bounded.

~~Since  $f_n$  is a sum of polynomials, it is continuous.~~

Also, since the partial sum of polynomials is a finite sum of polynomials, it is continuous.

Now it remains to show  $f_n$  uniformly converges to  $f$ .

Consider  $\epsilon > 0$

$$|f_n - f| = \left| \sum_{i=n+1}^{\infty} a_i x^i \right|$$

This is true since we can just choose  $N$  index of largest  $a_i$  value.

let  $z = \max\{|a_{n+1}|, |a_{n+2}|, \dots\}$  since  $\sum$  is bounded

$$\text{then } \left| \sum_{i=n+1}^{\infty} a_i x^i \right| < \left| \sum_{i=n+1}^{\infty} z x^i \right| = z \frac{|x|^{n+1}}{1-|x|}$$

So we can choose our  $n$  accordingly

So, by Thm,  $f$  is continuous.

~~make it~~  
 $= \frac{\epsilon}{z}$  not in terms of  $x$

so that inequality is satisfied

We can use the same Thm since  $\left| \sum_{n=1}^{\infty} n^{-2} x^n \right| \leq \sum_{n=1}^{\infty} |n^{-2} x^n| \leq \sum_{n=1}^{\infty} |n^{-2}| = \frac{\pi^2}{6}$

So since the sum is bounded,  
we can use the same  
Thm to say  $\sum_{n=1}^{\infty} n^{-2} x^n$  is  
continuous on  $[-1, 1]$

3) We first show that for  $0 < a < 1$ , we have uniform  
convergence on  $[-a, a]$

~~We know~~

Let  $M_n = a^n$   $\sum_{n=1}^{\infty} M_n$  converges and  $|x^n| \leq |a^n|$

so by Weierstrass M-test,

$\sum_{n=1}^{\infty} x^n$  converges uniformly on  $[-a, a]$  ( $|x| \leq a$ )

$\Rightarrow f(x)$  converges on  $(-1, 1)$

However consider

$|f_n(x) - f(x)| = \left| \sum_{k=n+1}^{\infty} x^k \right| = \frac{x^{n+1}}{1-x}$  which can get arbitrarily  
large as  $|x| \rightarrow 1$

So the ~~error~~  $N$  must depend  
on  $x$  to define  
the convergence

$\Rightarrow$  it is not uniform.