Math 104 HW9

(1) we know sum, ~~difference~~ quotient of infinitely differentiable functions are infinitely differentiable.

So, from Ex 3, $f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \le 0 \end{cases}$ is infinitely differentiable.

$f(1-x)$ is also infinitely differentiable, since it just reflects graph

Consider $g(x) = \dfrac{f(x)}{f(x)+f(1-x)}$. This is infinitely differentiable

and if $x \le 0$ $g(x) = 0$ if $x \ge 1$ $g(x) = 1$

and $g(x) \in [0,1]$ when $x \in (0,1)$ since numerator $\le$ denominator with both positive.

(2) Consider the polynomial

$$c_0 x + \frac{c_1}{2}x^2 + \frac{c_2}{3}x^3 + \dots + \frac{c_n}{n+1}x^{n+1} = f(x)$$

we know $f(0) = 0$ and $f(1)$ is given to be 0

Since f is a polynomial, ~~it~~ diff f' exists on $(0,1)$

So $\exists\, c \in (0,1)$ s.t. $f'(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n = 0$

as desired

(3) Since f' is continuous on $[a,b]$ for any $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|x-y| < \delta \Rightarrow |f'(x) - f'(y)| < \varepsilon$

So Now assume WLOG $x > t$. Then by MVT,

$\exists\, c$ ~~in~~ $\in (x,t)$ s.t. $f'(c) = \dfrac{f(t) - f(x)}{t - x}$

So there always exists such δ. $x < t$ is similar case.

④ Starting with $f(t) - f(\beta) = (t-\beta)\,Q(t)$

we differentiate $n-1$ times to get

$$f^{(n-1)}(t) = (n-1)\,Q^{(n-2)}(t) + Q^{(n-1)}(t)\,(t-\beta)$$

Now we can plug this into the original taylor expression

$$\Rightarrow P(\beta) = \sum_{k=1}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k + f(\alpha)$$

$$= \sum_{k=1}^{n-1} \frac{k\,Q^{(k-1)}(\alpha) + Q^{(k)}(\alpha)\,(\alpha-\beta)}{k!}\cdot(\beta-\alpha)^k + f(\alpha)$$

$$= \cancel{\sum_{k=0}^{n-1} \frac{Q^{(k-1)}(\alpha)}{(k-1)!}}$$

$$= \sum_{k=1}^{n-1} \frac{Q^{(k-1)}(\alpha)}{(k-1)!}(\beta-\alpha)^k + \sum_{k=1}^{n-1} \frac{Q^{(k)}(\alpha)\,(\alpha-\beta)^{k+1}}{k!} + f(\alpha)$$

$$= f(\beta) - \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n$$

$$\Rightarrow f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-a)^n$$

⑤

(a) If $f'(t) \neq 1$ for every real t, ~~either~~

Assume for contradiction that $f(a) = a$, $f(b) = b$

then $\exists\, c \in (a,b)$ s.t. $f'(c) = \dfrac{f(a)-f(b)}{a-b} = 1$

This contradicts $f'(t) \neq 1$ for every real t.

(b) $f'(t) = 1 + (-1)(1+e^t)^{-2}\cdot e^t$

$$= 1 - \frac{e^t}{(1+e^t)^2}$$ which is between 0 and 1

Also $f(t) = t + (1+e^t)^{-1}$

and since $(1+e^t)^{-1}$ always positive,

$f(t) > t \quad \forall t \in \mathbb{R}$

(c) By MVT, $\exists c \in (x_n, x_{n+1})$

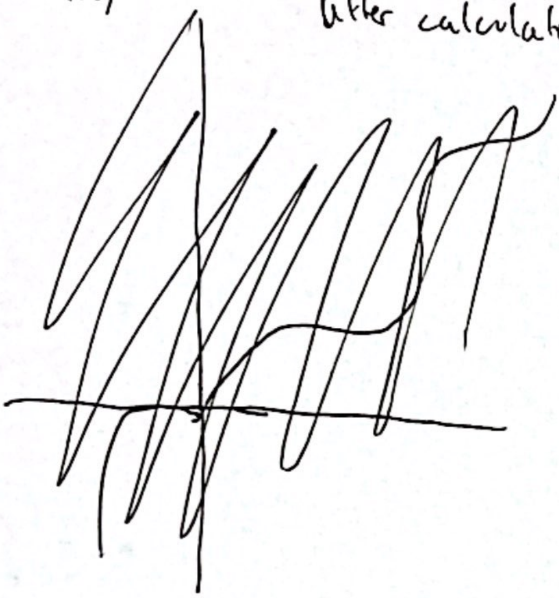
s.t. $f'(c) = \dfrac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} = \dfrac{f(x_{n+1}) - x_{n+1}}{f(x_n) - x_n}$

So the sequence $f(x_n) - x_n$ gets smaller and smaller

$\rightarrow 0 \Rightarrow$ at $x = \lim x_n$, $f(x) = x$ so

x is a fixed point.

(d)



After calculating $f(x_n) = x_{n+1}$, draw a line to the point (x_{n+1}, x_{n+1}) which is horizontal. Keep doing this and the function will converge at (x, x) where $x = \lim x_n$