104

12.10

If ~~[illegible]~~

S_n is bounded, $\exists M$ such that $M > |s_i|$ for all i

So, $\limsup |s_n| < |M| < +\infty$

If $\limsup |s_n| = M < +\infty$

then $\forall \varepsilon > 0$, $\exists N$ large enough such that $n > N \Rightarrow \left| \sup \{|s_n|, |s_{n+1}|, \ldots\} - M \right| < \varepsilon$

$\Rightarrow \sup \{|s_n|, |s_{n+1}|, \ldots\} < M + \varepsilon$

$\sup \{|s_n|, |s_{n+1}|, \ldots\} > M - \varepsilon$

Now consider the upper bound as $\max(M+\varepsilon, |S_1|, |S_2|, \ldots |S_{n-1}|) + 1$

and the lower bound as $\min(-M-\varepsilon, -|S_1|, -|S_2|, \ldots, -|S_{n-1}|) - 1$

These are upper bounds and lower bounds respectively of S_n.

by definition of sup. So S_n is bounded

12.12

(a) Clearly $\liminf \sigma_n \leq \limsup \sigma_n$

It remains to show $\limsup \sigma_n \leq \limsup S_n$ and $\liminf S_n \leq \liminf \sigma_n$

(b) let $\limsup S_n = A$

then for all $\varepsilon > 0$ $\exists N$ such that $n > N \Rightarrow |A - \sup\{s_n, s_{n+1}, \ldots\}| < \varepsilon$

Since sup is non increasing

Now consider σ_n for $n > N$ $\quad (\Rightarrow)$ $\sup\{s_n, s_{n+1}, \ldots\} < A + \varepsilon$

$\Rightarrow \sigma_n = \frac{1}{n}(s_1 + s_2 + \ldots + s_n) = \frac{1}{n}\left((s_1+s_2+\ldots+s_N) + (s_{N+1}+\ldots+s_n)\right) < \frac{1}{n}(s_1+\ldots+s_N) + \frac{1}{n}(n-N)(A+\varepsilon)$

$\leq \frac{1}{n}(s_1+s_2+\ldots+s_N) + \frac{1}{n}(n-N)(A+\varepsilon)$

So now considering $\limsup\limits_{n\to\infty}$ we have $\limsup \sigma_n \leq (A+\varepsilon) \Rightarrow \limsup \sigma_n \leq \limsup S_n$

We can do the same thing for lim inf.

(b) if $\lim S_n$ exists, $\liminf S_n = \limsup S_n = A$

$\Rightarrow \liminf \sigma_n = \limsup \sigma_n = A$

since the sequences σ lie in between S

(c)

$S_i = 1$ if i is even

0 if i is odd

14.2

(a) $= \sum \frac{1}{n} - \frac{1}{n^2}$

$= \sum \frac{1}{n} - \sum \frac{1}{n^2}$ doesnt converge by p-series

~~which doesnt converge~~

(b) Does not converge since it oscillates between $-1, 0$

(c) Converges by p-series test $p=2$

(d) Converges by ratio test

lim sup = lim inf $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^3}{n^3 \cdot 3} \right| = \frac{1}{3} < 1$

(e) Converges by ratio test

lim sup = lim inf $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!}{(n+1)n^2} \right| = 0 < 1$

(f) $\sum_{n=1}^{\infty} \frac{1}{n^n} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^n} < 1 + \sum_{n=2}^{\infty} \frac{1}{n^2}$

which converges by comparison and p-series test $p=2$

(g) Converges by ratio test $\lim$ (lim sup = lim inf) $\left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{n+1}{2n}\right| = \frac{1}{2}$

14.10

If n is even $a_n = 2^n$ $2^{-1}, 2^2, 2^0, 2^3, 2^1, 2^4, \dots$

If n is odd $a_n = 2^{n-2}$ Here, $\liminf \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}$

$\limsup \left|\frac{a_{n+1}}{a_n}\right| = 8$

while root test

$\limsup |a_n|^{1/n} > 1$

$= 2$

Rudin

Ch 3 #6

(a) $a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \geq \frac{1}{2\sqrt{n+1}}$ which diverges by p-series test $p = 0.5$

(b) $a_n = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} < \frac{1}{n(\sqrt{n}+\sqrt{n})} = \frac{1}{2n^{1.5}}$ which converges by p-series test $p = 1.5$

(c) $\limsup |a_n|^{1/n} = \limsup_{n\to\infty} (\sqrt[n]{n} - 1) = 0 < 1$ which converges by root test.

(d) If $|z| \leq 1$ then the series doesn't pass sanity test as $\lim \frac{1}{1+z^n}$ doesn't go to 0.

If $|z| > 1$ then ratio test

$\Rightarrow \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{1+z^{n+1}}{1+z^n}\right| = \lim \left|\frac{z^{n+1}}{z^n}\right| = |z| > 1 \Rightarrow$ divergent

#7

Consider the set S of n such that $a_n < \frac{1}{n^2} \Rightarrow \frac{\sqrt{a_n}}{n} < \frac{1}{n^2}$

$\Rightarrow \sum \frac{\sqrt{a_n}}{n} < \sum \frac{1}{n^2}$ which converges.

Consider the set T of n such that $a_n \geq \frac{1}{n^2} \Rightarrow \frac{\sqrt{a_n}}{n} \leq a_n \Rightarrow \sum \frac{\sqrt{a_n}}{n} \leq \sum a_n$ which converges

So adding these 2 gives an upper bound which converges.

#9

(a) By root test $\limsup |n^3 z^n|^{1/n} = \limsup |n^{3/n}| |z| = |z| \cancel{<1}$

So radius of convergence is $\boxed{1}$

(b) By ratio test $\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left| \frac{\frac{2^{n+1}}{(n+1)!} z^{n+1}}{\frac{2^n}{n!} z^n} \right| = |z| \lim \left|\frac{2}{n+1}\right| = 0 \cancel{<} 1$

So radius of convergence is $\boxed{+\infty}$

(c) By root test $\limsup \left|\frac{2^n z^n}{n^2}\right|^{1/n} = \limsup \left|\frac{2z}{n^{2/n}}\right| = |2z| < 1$

$\Rightarrow |z| < \frac{1}{2}$

radius of convergence is $\boxed{\frac{1}{2}}$

(d) By root test $\limsup \left|\frac{n^3 z^n}{3^n}\right|^{1/n} = \limsup |n^{3/n}| \left|\frac{z}{3}\right| \Rightarrow \left|\frac{z}{3}\right| < 1$

$\Rightarrow |z| < 3$

radius of convergence $= \boxed{3}$

#11

(a) $= \Sigma\left(1 - \frac{a_n}{a_{n+1}}\right)$

(b)