Math 104

(1) Consider a sequence p_n in K

$$p_1, p_2, p_3, \dots \qquad p_i \in [0,1]\times[0,1]$$

let $p_i = (a_i, b_i)$

We know a_i is bounded so it has a convergent subsequence. Consider the terms in p_i that correspond to this subsequence of a_i's, call it $q_i = (a_i, b_i)$ such that a_i converges.

Now since the new b_i is bounded, it also must have a convergent subsequence. Consider the terms in q_i that correspond to this subsequence of b_i's. Call it $r_i = (a_i, b_i)$ where $a_i \to a$ and $b_i \to b$

since all subsequences of a convergent sequence converge to the limit.

So

Since $r_i \to (a,b) \in [0,1]\times[0,1]$ ~~...~~

we have that $[0,1]^2$ in $\mathbb{R}^2$ is sequentially compact.

(2)

Assume for the sake of contradiction that E is countable.

E

1	4 7 4 4 7 7
2	7 7 4 7 4 7
3	7 4 4 7 7 7
4	7 7 7 7 7 7
⋮	

Consider the decimal $0.7474\ldots$ where each digit is the opposite of the one on the diagonal

This number is not listed $\Rightarrow$ contradiction

so E is not diagonal.

We know E is bounded since it is fully contained in $[0,1]$

~~Consider~~ Now we just have to show it is closed

Consider some subsequence that is convergent ~~to the same limit~~

$s_1, s_2, s_3, \ldots$ where $s_i \in E$

Consider the k^{th} digit ~~of~~ of each element of this sequence. Since it is convergent there is some N such that $n > N \Rightarrow s_n$ has the same k^{th} digit as the k^{th} digit of the limit. This is ~~~~ because we can consider some $\varepsilon = 10^{-k-1}$

~~~~ Since s_n ~~~~ always has a k^{th} digit of 4 or 7 the limit must have all digits be 4 or 7. $\Rightarrow$ it is contained in E.

Therefore E is closed and bounded $\Rightarrow$ compact.

(3)

If there are a finite number of A_i then this is not true. Otherwise we can consider an enumeration of $\mathbb{Q}$ since it is countable, and let A_i be ~~~~ a set of just the i^{th} rational number

$B = \bigcup_i A_i = \mathbb{Q} \Rightarrow \bar{B} = \mathbb{R}$

However $\bigcup_i \bar{A_i} = \bigcup_i A_i = \mathbb{Q}$

clearly there are elements of $\mathbb{R}$ not in $\mathbb{Q}$ so this is a strict inclusion.

(4)

In question 2 $E = x \in [0,1]$ s.t. decimal expansion only ~~4~~4's and 7's is closed. However, it is not a countable union of closed intervals, since ~~A~~ no ~~nonempty interval~~ nontrivial intervals are contained in E ~~~~ and E is not countable.
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