Math 104 HW1

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January 23, 2022

1 Problems

1.10: Let P_n be the proposition that $(2n+1)+(2n+3)+(2n+5)+\cdots+(4n-1) = 3n^2$. We proceed with induction. Our base case is P_1:

$$2 \cdot 1 + 1 = 3 = 3 \cdot 1^2$$

which is true. Now, assume P_n:

$$(2n+1) + (2n+3) + (2n+5) + \cdots + (4n-1) = 3n^2$$

We can add $(4n+1) + (4n+3) - (2n+1)$ to both sides to get

$$(2n+3) + (2n+5) + \cdots + (4n+3) = 3n^2 + 6n + 3$$

$$\implies (2(n+1)+1) + (2(n+1)+3) + \cdots + (4(n+1)-1) = 3(n+1)^2$$

which is precisely P_{n+1}. So, our induction is complete.

1.12: (a)

$$(a+b)^1 = a + b = \binom{1}{0}a + \binom{1}{1}b$$

$$(a+b)^2 = a^2 + 2ab + b^2 = \binom{2}{0}a^2 + \binom{2}{1}ab + \binom{2}{2}b^2$$

$$(a+b)^3 = (a+b)(a^2+2ab+b^2) = a^3 + 3a^2b + 3ab^2 + b^3$$

$$= \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$$

(b)

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}$$

$$= \frac{n!(n-k+1)}{k!(n-k+1)!} + \frac{n!k}{k!(n-k+1)!} = \frac{n!(n+1)}{k!(n-k+1)!}$$

$$= \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}$$

(c) Let P_n be the proposition that

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

We proved the base case in part (a). Now assuming P_n, we can multiply by $(a+b)$ on both sides:

$$(a+b)^{n+1} = (a+b)\sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} = \sum_{i=0}^{n} \binom{n}{i} a^{i+1} b^{n-i} + \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i+1}$$

$$= \sum_{i=0}^{n} \left(\binom{n+1}{i+1} - \binom{n}{i+1} \right) a^{i+1} b^{n-i} + \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i+1}$$

$$= \sum_{i=1}^{n+1} \left(\binom{n+1}{i} - \binom{n}{i} \right) a^i b^{n-i+1} + \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i+1}$$

$$= a^{n+1} + \sum_{i=1}^{n} \left(\binom{n+1}{i} - \binom{n}{i} \right) a^i b^{n-i+1} + b^{n+1} + \sum_{i=1}^{n} \binom{n}{i} a^i b^{n-i+1}$$

$$= a^{n+1} + \left(\sum_{i=1}^{n} \binom{n+1}{i} a^i b^{n-i+1} \right) + b^{n+1}$$

$$= \sum_{i=0}^{n+1} \binom{n+1}{i} a^i b^{n-i+1}$$

which is precisely P_{n+1}, so our induction is complete.

2.1: Consider the polynomial $x^2 - 3$. By the Rational Zero's theorem, the only possible rational zeros of this polynomial are $3, -3, 1, -1$. However, $\sqrt{3}$ is a zero of this polynomial, therefore it must be irrational.

Consider the polynomial $x^2 - 5$. By the Rational Zero's theorem, the only possible rational zeros of this polynomial are $5, -5, 1, -1$. However, $\sqrt{5}$ is a zero of this polynomial, therefore it must be irrational.

Consider the polynomial $x^2 - 7$. By the Rational Zero's theorem, the only possible rational zeros of this polynomial are $7, -7, 1, -1$. However, $\sqrt{7}$ is a zero of this polynomial, therefore it must be irrational.

Consider the polynomial $x^2 - 24$. By the Rational Zero's theorem, the only possible rational zeros of this polynomial are $24, -24, 12, -12, 8, -8, 6, -6, 4, -4, 3, -3, 2, -2, 1, -1$. However, $\sqrt{24}$ is a zero of this polynomial, therefore it must be irrational.

Consider the polynomial $x^2 - 31$. By the Rational Zero's theorem, the only possible rational zeros of this polynomial are $31, -31, 1, -1$ However, $\sqrt{31}$ is a zero of this polynomial, therefore it must be irrational.

2.2: Consider the polynomial $x^3 - 2$. By the Rational Zero's theorem, the only possible rational zeros of this polynomial are $2, -2, 1, -1$. However, $\sqrt[3]{2}$ is a zero of this polynomial, therefore it must be irrational.

Consider the polynomial $x^7 - 5$. By the Rational Zero's theorem, the only possible rational zeros of this polynomial are $5, -5, 1, -1$. However, $\sqrt[7]{5}$ is a zero of this polynomial, therefore it must be irrational.

Consider the polynomial $x^4 - 13$. By the Rational Zero's theorem, the only possible rational zeros of this polynomial are $13, -13, 1, -1$. However, $\sqrt[4]{13}$ is a zero of this polynomial, therefore it must be irrational.

2.7: (a) Notice that $\sqrt{4+2\sqrt{3}} - \sqrt{3} = \sqrt{1+2\sqrt{3}+3} - \sqrt{3} = \sqrt{(1+\sqrt{3})^2} = 1$, which is rational.
(b) Notice that $\sqrt{6+4\sqrt{2}} - \sqrt{2} = \sqrt{4+4\sqrt{2}+2} - \sqrt{2} = \sqrt{(2+\sqrt{2})^2} - \sqrt{2} = 2$, which is rational.

3.6: By the triangle inequality, we know that $|a+b| \leq |a| + |b|$ for all $a, b \in R$. Since $a + b \in R$, we can also say by the triangle inequality $|(a+b)+c| \leq |a+b| + |c|$ for all $a, b, c \in R$. So, we can plug the first inequality back in to get $|a+b+c| \leq |a| + |b| + |c|$.

4.11: Assume for the sake of contradiction that there were finitely many rationals between $a, b \in R$. Let there be k rationals. Let's order them such that

$$a \leq x_1 < x_2 < \cdots < x_k \leq b$$

By the denseness of Q, we know there exists a rational between any two reals. So, there exists another rational number between x_1 and x_2 since $x_1, x_2 \in R$. This means there are at least $k+1$ rationals between a and b, which is a contradiction to the fact that there were k rationals. So, there must be infinitely many rational numbers between a and b.

4.14: (a) We know that for all $a \in A$, $a \leq \sup A$. We know that for all $b \in B$, $b \leq \sup B$. So, we can add these to get $a + b \leq \sup A + \sup B$. This means that $\sup A + \sup B$ is an upper bound, and it remains to show that it is the least upper bound. Assume for the sake of contradiction that the actual upper bound is $\sup A + \sup B - \epsilon$ for some $\epsilon > 0$. We know there exists an element $a_1 \in A$ such that $a_1 > \sup A - \frac{\epsilon}{2}$, since $\sup A$ is the least upper bound. Similarly we know there exists an element $b_1 \in B$ such that $b_1 > \sup B - \frac{\epsilon}{2}$. So, adding these gives us $a_1 + b_1 > \sup A + \sup B - \epsilon$. However, this contradicts the fact that the upper bound of the set was $\sup A + \sup B - \epsilon$. Therefore the least upper bound of this set must be $\sup A + \sup B$.

7.5: (a) We know $(\sqrt{n^2+1} - n)(\sqrt{n^2+1} + n) = 1 \implies \sqrt{n^2+1} - n = \frac{1}{\sqrt{n^2+1}+n}$. Since the denominator goes to infinity, our desired limit is 0.
(b) We know $(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n) = n \implies \sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$. Here, the numerator goes to 1 and the denominator goes to 2, so our desired answer is $\frac{1}{2}$.
(c) We know $(\sqrt{4n^2+n} - 2n)(\sqrt{4n^2+n} + 2n) = n \implies \sqrt{4n^2+n} - 2n = \frac{n}{\sqrt{4n^2+n}+2n} = \frac{1}{\sqrt{4+\frac{1}{n}}+2}$. Here, the numerator goes to 1 and the denominator goes to 4, so our desired answer is $\frac{1}{4}$.