Math 104 HW2

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1 Problems

9.9:

(a) We know that for all M, $\exists$ N such that $n > N \implies s_n > M$. This also means $n > \max(N, N_0) \implies t_n \geq s_n > M$. So, for all M, we have $n > \max(N, N_0) \implies t_n > M$, which means $\lim t_n = +\infty$.

(b) We know that for all M, $\exists$ N such that $n > N \implies t_n < M$. This also means $n > \max(N, N_0) \implies s_n \leq t_n < M$. So, for all M, we have $n > \max(N, N_0) \implies s_n < M$, which means $\lim s_n = -\infty$.

(c) Assume for the sake of contradiction that $\lim s_n > \lim t_n$. Let $\lim s_n = s$ and $\lim t_n = t$. We know for all $\epsilon_s > 0$, there exists an N_s such that $n > N_s \implies |s_n - s| < \epsilon_s$. Similarly, for all $\epsilon_t > 0$, there exists an N_t such that $n > N_t \implies |t_n - t| < \epsilon_t$.Consider $\epsilon_s = \frac{s-t}{2}$. We see that there exists an N such that $n > N \implies |s_n - s| < \frac{s-t}{2} \implies s_n > t + \frac{s-t}{2}$. This means that $n > \max(N_s, N_0) \implies s_n > t + \frac{s-t}{2} \implies t_n > t + \frac{s-t}{2} \implies |t_n - t| > \frac{s-t}{2}$. This means $n > \max(N_0, N_s, N_t) \implies |t_n - t| > \frac{s-t}{2}$. However, choosing $\epsilon_t = \frac{s-t}{2}$ this contradicts the limit definition of t. So, it must be true that $\lim s_n \leq \lim t_n$.

9.15: We want to show that for all $\epsilon > 0$, $\exists$ N such that $n > N \implies |\frac{a^n}{n!}| < \epsilon$. Consider an integer $N > \frac{a}{\epsilon} \cdot \left|\frac{a^b}{b!}\right|$, where b be some integer greater than $|a|$. We can break this inequality into two parts:

$$\left|\frac{a^n}{n!}\right| = \left|\frac{a^b}{b!}\right| \left|\frac{a^{n-b}}{(b+1)(b+2)\cdots n}\right| = \left|\frac{a^b}{b!}\right| \left|\frac{a}{b+1}\right| \left|\frac{a}{b+2}\right| \cdots \left|\frac{a}{n}\right|$$

$$< \left|\frac{a^b}{b!}\right| \left|\frac{a}{n}\right| < \left|\frac{a^b}{b!}\right| \frac{|a|}{\left|\frac{a^b}{b!}\right| \cdot \frac{a}{\epsilon}} = \epsilon$$

So, we have found that there always exists an N in terms of ϵ, meaning $\lim \left|\frac{a^n}{n!}\right| = +\infty$

10.7: By definition, we know that for all $\epsilon > 0$, there exists an element $s \in S$ such that $s > \sup S - \epsilon$. So consider the sequence (s_n) where s_i is a randomly selected element from all the elements $s \in S$ such that $s > \sup S - \frac{\epsilon}{2^i}$. Since we know $\sup S \notin S$, we have

$$\sup S > s_i > \sup S - \frac{\epsilon}{2^i}$$

Now, by the squeeze theorem, we can say $\lim s_i = \sup S$, since the limit of both sides are also $\sup S$.

10.8: Consider

$$\sigma_{n+1} - \sigma_n = \frac{1}{n+1}(s_1 + s_2 + \cdots + s_n) + \frac{s_{n+1}}{n+1} - \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$$
$$= \frac{s_{n+1}}{n+1} - \frac{s_1 + s_2 + \cdots + s_n}{n(n+1)} \geq \frac{s_{n+1}}{n+1} - \frac{ns_n}{n(n+1)} = \frac{s_{n+1} - s_n}{n+1} > 0$$

Therefore, the sequence (σ_n) is increasing.

10.9:

(a) $s_2 = \frac{1}{2} \cdot 1^2 = \frac{1}{2}$, $s_3 = \frac{2}{3} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{6}$, $s_4 = \frac{3}{4} \cdot \left(\frac{1}{6}\right)^2 = \frac{1}{48}$.

(b) I claim that $0 \leq s_n \leq 1$ for all n. Our base case is $n = 1$, which works since $s_1 = 1$. For the inductive step, assume $0 \leq s_n \leq 1$. We have $s_{n+1} = \frac{n}{n+1}s_n^2$. Since $0 \leq \frac{n}{n+1} \leq 1$ and $0 \leq s_n \leq 1 \implies 0 \leq s_n^2 \leq 1$. So, combining these gives $0 \leq s_{n+1} \leq 1$. This completes our induction. Also since $s_n^2 \leq s_n$ and $\frac{n}{n+1} < 1$, we can multiply these and get that $s_{n+1} < s_n$. Now, since we know the sequence is bounded and monotone, it is convergent, meaning its limit exists.

(c) Since we know that $s_n \leq 1$, we have $s_{n+1} \leq \frac{n}{n+1}s_n$. Now, I claim that $s_n \leq \frac{1}{n}$. Our base case is $n = 1$, which works since $s_1 = 1$. For the inductive step, assume $s_n \leq \frac{1}{n}$. We know that $s_{n+1} \leq \frac{n}{n+1}s_n$. We can substitute to get $s_{n+1} \leq \frac{1}{n+1}$ as desired. So now we know that $0 \leq s_n \leq \frac{1}{n}$. We can use the squeeze theorem and see that $\lim s_n = 0$, since both its upper and lower bound go to 0.

10.10:

(a) $s_2 = \frac{1}{3} \cdot (1 + 1) = \frac{2}{3}$, $s_3 = \frac{1}{3} \cdot (1 + \frac{2}{3}) = \frac{5}{9}$, $s_4 = \frac{1}{3} \cdot (1 + \frac{5}{9}) = \frac{14}{27}$.

(b) Our base case is $s_1 = 1 > \frac{1}{2}$. For the inductive step, assume $s_n > \frac{1}{2}$. We know that $s_{n+1} = \frac{1}{3}(1 + s_n) > \frac{1}{3}(1 + \frac{1}{2}) = \frac{1}{2}$, as desired.

(c) $s_{n+1} - s_n = \frac{1}{3}(s_n + 1) - s_n = \frac{1}{3} - \frac{2}{3}n < \frac{1}{3} - \frac{2}{3} \cdot \frac{1}{2} = 0$. So, our sequence is decreasing.

(d) Since s_n is decreasing and $1 \geq s_n > \frac{1}{2}$, it is bounded and monotone. This implies it converges, meaning $\lim s_n$ exists. Since the limit exists, we can take the limit of both sides of the recurrence relation and see that $\lim s_n = \frac{1}{3}(\lim s_n + 1) \implies \lim s_n = \frac{1}{2}$.

10.11:

(a) Since $1 \geq 1 - \frac{1}{4n^2} \geq 0$ for all $n \geq 1$, we know that $0 \leq t_n \leq 1$. Also, we know that $\frac{t_{n+1}}{t_n} = 1 - \frac{1}{4n^2} < 1$, so t_n is decreasing. Since the sequence is monotone and bounded, it is convergent and $\lim t_n$ exists.

(b) Around 0.5? It seems to converge above 0 since the terms get really close to 1 really fast.

Squeeze Theorem: By part 9.9(c), where $N_0 = 0$, we know that $\lim b_n \leq \lim c_n = L$. Also, by part 9.9($c$), where $N_0 = 0$, we know that $\lim a_n = L \leq \lim b_n$. Combining these gives $\lim b_n = L$.