Math 104 HW3

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1 Problems

10.6:

(a) Consider two indices n and m such that $m > n$.

$$|s_m - s_n| = |s_m - s_{m-1} + s_{m-1} - s_{m-2} + \cdots + s_{n+1} - s_n|$$
$$\leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \cdots + |s_{n+1} - s_n|$$
$$\leq 2^{-(m-1)} + 2^{-(m-2)} + \cdots + 2^{-n}$$
$$\leq 2^{-m}$$

Consider $\epsilon > 0$. For an integer $N > -\log_2 \epsilon$, which exists due to the archimedian principle, we have $n, m > N \implies |s_m - s_n| \leq 2^{-m} < 2^{\log_2 \epsilon} = \epsilon$. So, the sequence is cauchy and convergent.

(b) No, we can construct a sequence $s_n = \sum_{i=1}^{n} \frac{1}{2n}$. This satisfies the property that $|s_n - s_{n+1}| = \frac{1}{2n} < \frac{1}{n}$, but it is divergent.

11.2:

(a) (a) Consider the sub-sequence $\{a_{2n}\}_{n=1}^{\infty}$. This has the property that $a_{2i} = 1 = a_{2(i+1)}$. So, it is monotone.

(b) The sequence $\{b_n\}_{n=1}^{\infty}$ is monotone in itself.

(c) The sequence $\{c_n\}_{n=1}^{\infty}$ is monotone in itself.

(d) The subsequence $\{d_n\}_{n=1}^{\infty}$ is monotone. We can check this by showing $d_{n+1} \leq d_n$ for $n \geq 1$.

$$d_{n+1} - d_n = \frac{6(n+1)+4}{7(n+1)-3} - \frac{6n+4}{7n-3} = \frac{6n+10}{7n+4} - \frac{6n+4}{7n-3} = \frac{(6n+10)(7n-3) - (7n+4)(6n+4)}{(7n+4)(7n-3)}$$

$$= \frac{-46}{(7n+4)(7n-3)} < 0 \implies d_{n+1} \leq d_n$$

(b) (a) $\{1, -1\}$
  (b) $\{0\}$
  (c) $\{+\infty\}$
  (d) $\{\frac{6}{7}\}$

(c) (a) $\limsup a_n = 1$, $\liminf a_n = -1$
  (b) $\limsup b_n = 0$, $\liminf b_n = 0$
  (c) $\limsup c_n = +\infty$, $\liminf c_n = +\infty$
  (d) $\limsup d_n = \frac{6}{7}$, $\liminf d_n = \frac{6}{7}$

(d) (a) Does not converge or diverge.
  (b) Converges to 0.
  (c) Diverges to $+\infty$
  (d) Converges to $\frac{6}{7}$

(e) $(a_n), (b_n), (d_n)$ are bounded

11.3:

(a) (a) Consider the sub-sequence $\{s_{6n}\}_{n=1}^{\infty}$. This has the property that $s_{6i} = \cos(2\pi i) = 1 = \cos(2\pi(i+1)) = s_{6(i+1)}$. So, it is monotone.
  (b) The sequence $\{t_n\}_{n=1}^{\infty}$ is monotone in itself. To show this,

$$t_{n+1} - t_n = \frac{3}{4(n+1)+1} - \frac{3}{4n+1} = \frac{3}{4n+5} - \frac{3}{4n+1} = \frac{-12}{(4n+5)(4n+1)} < 0$$

$$\implies t_{n+1} \leq t_n$$

  (c) Consider the sub-sequence $\{u_{2n}\}_{n=1}^{\infty}$. This has the property that $u_{2i} = \left(-\frac{1}{2}\right)^{2i} = (\frac{1}{4})^i \geq (\frac{1}{4})^{i+1} = \left(-\frac{1}{2}\right)^{2i+2} = u_{2(i+1)}$. So, it is monotone.
  (d) Consider the sub-sequence $\{v_{2n}\}_{n=1}^{\infty}$. This has the property that $v_{2i} = 1 + \frac{1}{2i} \geq 1 + \frac{1}{2i+2} = v_{2i+2}$. So, it is monotone.

(b) (a) $\{1, \frac{1}{2}, -\frac{1}{2}, -1\}$
  (b) $\{0\}$
  (c) $\{0\}$
  (d) $\{-1, 1\}$

(c) (a) $\limsup s_n = 1$, $\liminf s_n = -1$
  (b) $\limsup t_n = 0$, $\liminf t_n = 0$
  (c) $\limsup u_n = 0$, $\liminf u_n = 0$
  (d) $\limsup v_n = 1$, $\liminf v_n = -1$

(d) (a) Does not converge or diverge.
    (b) Converges to 0.
    (c) Converges to 0
    (d) Does not converge or diverge.

(e) $(s_n), (t_n), (u_n), (v_n)$ are bounded

11.5:

(a) All real $x \in [0, 1]$. We can create a sequence of rationals whose limit is x using the dedekind cut. For every $\epsilon > 0$, there exists a rational thats between x and $x - \epsilon$.

(b) $\limsup q_n = 1, \liminf q_n = 0$

limsup: $\limsup$ works on sequences, taking the limit of the sup of tail subsets. On the other hand, sup works on sets. One big difference is that $\limsup$ doesn't care about the first finite terms of the sequence. We can just throw them away since they don't matter in the limit of the tail. However, they do matter in the sup, as it is an upperbound for the whole set. One example of an incorrect statement is that $\limsup$ of a sequence is larger than every element in the sequence.