HOMEWORK #2

9.9, 9.15, 10.7, 10.8

Ab recursive seq: 10.9, 10.10, 10.11

Squeeze Test. Let a_n, b_n, c_n be 3 sequences st $a_n \leq b_n \leq c_n$?
$L = \lim a_n = \lim c_n$. Show that $b_n = L$.

9.9 Suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$.

(c) Prove that if $\lim s_n$ and $\lim t_n$ exist, then $\lim s_n \leq \lim t_n$

$s_n - s < \varepsilon$

- $s_n \leq t_n$ for all $n > N_0$
- Prove $\lim s_n \wedge \lim t_n \rightarrow \lim s_n \leq \lim t_n$
- Let $\lim s_n = s$ and $\lim t_n = t$
- Assume that $\lim s_n$ and t_n exist,
- Need to prove $s \leq t$ when $s_n \leq t_n$ for all $n > N_0$
- $s_n \leq t_n$ implies $t_n - s_n \geq 0$ for all $n > N_0$
- $s \leq t$ implies $t - s \geq 0$
- Because $t_n - s_n \geq 0$, then $\lim(t_n - s_n) \geq 0$, which implies $\lim t_n - \lim s_n \geq 0 = t - s \geq 0 = t \geq s$ $= \lim t_n \geq \lim s_n = \lim s_n \leq \lim t_n$ □

□

9.15 Show $\lim_{n\to\infty} \frac{a^n}{n!}$ for all $a \in \mathbb{R}$.

- $s_n = \frac{a^n}{n!}$ $n=1$ $s_n = \frac{a}{1}$, $n=2$ $s_n = \frac{a^2}{2}$, $n=3$ $s_n = \frac{a^3}{6}$, $n=4$ $s_n = \frac{a^4}{24}$
- $s_n = \dfrac{\frac{a}{n}}{\frac{n(n+1)!}{n}} = \dfrac{\frac{a}{n}}{\frac{(n+1)!}{1}} = \dfrac{a}{n(n+1)!}$ $\lim_{n\to\infty} \dfrac{a}{n(n+1)!} = 0$ *using 9.7 (a)

□

10.7 Let S be a bounded nonempty subset of $\mathbb{R}$ such that $\sup S$ is not in S. Prove there is a sequence (s_n) of points in S such that $\lim s_n = \sup S$. (See 11.11)

def 9 pg 45
- The set S is said to be bounded if it is bounded above and bounded below. Thus S is bounded if [4.2 (a) pg 21 Ross]

(a)
- there exist real #'s m and M such that $S \subseteq [m, M]$.
- If a real # M satisfies $s \leq M$ for all $s \in S$, then M is an upper bound of S. (a)
- Example 3 on pg 22 of Ross, If a set S has a maximum, then $\max S = \sup S$.

- we're given that $\sup S \notin S$, this implies that ~~there exist~~ real numbers m and M such that (m, M) ~~$\{s \in \mathbb{R} : m < s < M\}$~~ $\forall s \in S$ ~~we have~~ $m < s < M$
- ~~this implies that M is the least upper bound of S and the inequality ensures that $\sup S \notin S$.~~

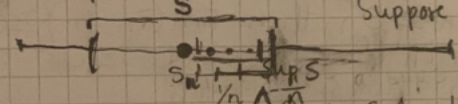
- ~~Let s_n be a sequence of points in S.~~

Not sure how to show that $\lim s_n = \sup S$ using M

want to prove $\lim s_n = \sup S$
for this to happen we need to impose 2 conditions
(1) s_n is increasing
(2) $\lim s_n = \sup S$

- Let $A = \sup S$ we want $A - \frac{1}{n} < s_n < A$

A = least upper bound of s_n



Suppose $\exists\, n_0$ st. we can't find s_{n_0}

for all $s \in S$, we have $A - \frac{1}{n} \geq s$, thus $A - \frac{1}{n}$ would be an upper bound for S.

10.8 Let (s_n) be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. Prove ($\sigma_n$ is an increasing sequence).

- s_n is an increasing sequence $\Rightarrow s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$ it's monotone.
- $\sigma_n = \frac{s_1}{n} + \frac{s_2}{n} + \cdots + \frac{s_n}{n}$
- ~~Suppose for the sake of contradiction σ_n is a decreasing sequence. This implies that $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$. Because the elements of σ_n are fractions a/b where $b \neq 0$, as $n \to \infty$, b (the denominator) will get larger and larger. The only way σ could be decreasing is if the numerators $(s_1 + s_2 + \ldots + s_n)$ are decreasing as $n \to \infty$; however, we know s_n is an increasing sequence of positive numbers, as given by the problem, so we reach a contradiction.~~

Does the case where σ_n is not monotone apply? Was it proved implicitly in this proof? yes

□

$\sigma_{n+1} > \sigma_n \Leftrightarrow s_{n+1} > \sigma_n$

$\{\underbrace{1, 2, 3}, 4\}$

$n \cdot s_{n+1} \geq s_1 + s_2 + \cdots + s_n$

$\underbrace{s_{n+1} + s_{n+1} + \cdots + s_{n+1}}_{n \text{ times}}$

$\geq \underbrace{s_1 + s_2 + \cdots + s_n}_{n \text{ terms}}$

$\frac{s_1 + s_2}{2} \leq \frac{s_1 + s_2 + s_3}{3}$

$\Leftrightarrow \quad s_3 \geq \frac{s_1 + s_2}{2}$

$\boxed{10.9}$ Let $s_1 = 1$ and $s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2$ for $n \geq 1$.

(a) Find s_2, s_3, and s_4.

$s_2 = \left(\frac{1}{2}\right) \cdot 1 = \frac{1}{2}$ / $s_3 = \frac{2}{3} \cdot \frac{1}{4} = \frac{2}{12} = \frac{1}{6}$ / $s_4 = \frac{3}{4} \cdot \frac{1}{24} = \frac{1}{48}$ $\frac{1}{12}$

(b) Show $\lim s_n$ exists.

We observe from (a) that s_n is getting smaller for greater n and seems to be approaching 0. So we will show the limit exists by first proving $0 < s_{n+1} < s_n \leq 1$ for all $n \geq 1$.
To show $s_n \leq 1$, we will procede with induction:
Base Case: $s_1 = 1$
Induction step: s_{n+2}

$s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2 \Rightarrow \underset{\text{to } n\text{'s}}{\overset{\text{add}}{+1}} \Rightarrow s_{n+2} = \left(\frac{n+1}{n+2}\right) s_{n+1}^2$

$\Rightarrow s_{n+2} = \left(\frac{n+1}{n+2} s_{n+1}\right) s_{n+1}$, We observe here that $n+1/n+2 < 1$ because $n+2 > n+1$ for any n. Under our assumption, $s_{n+1} < 1$. So the product of $\left(\frac{n+1}{n+2}\right)(s_{n+1})(s_{n+1}) < 1$. Because $s_{n+1} > 0$, $s_{n+2} < 0$ therefore we get:

$$0 < s_{n+2} < s_{n+1} \leq 1 \qquad (1)$$

As n increases, s_n becomes smaller than 1 and smaller than the previous term in the sequence. So line (1) tells us that s_n is a bounded monotone decreasing sequence $\Rightarrow$ therefore it is convergent by Theorem 10.2 on pg 57 of Ross: "All bounded monotone sequences converge." $\Rightarrow$ therefore the lim of s_n exists.

(c) Prove $\lim s_n = 0$.
Let us assume for the sake of contradiction that the $\lim s_n \neq 0$ and that $\lim s_n = s$ (some number s, since we know s_n converges by part (b)). Then we know $s = \lim_{n+1} = \lim \frac{n}{n+1} s_n^2 = \lim \frac{n}{n+1} \cdot \lim s_n^2$.