ALEX
TORRES
HOMEWORK 1
1.10, 1.12, 2.1, 2.2, 2.7, 3.6, 4.11, 4.14, 7.5

1.10

Prove $(2n+1)+(2n+3)+(2n+5)+\ldots+(4n-1)=3n^2$ for all positive integers n.

- The nth proposition is P_n: "$(2n+1)+(2n+3)+(2n+5)+\ldots+(4n-1)=3n^2$."
- Base Case $n=1$: $P_1 = 2(1)+1 = 3(1)^2 \Rightarrow 3=3$ ✓
- Induction Step: Suppose $(2n+1)+(2n+3)+(2n+5)+\ldots+(4n-1)=3n^2$ true.
  - want to prove P_{n+1} is true, add $n+1$ to both sides:

$$\underbrace{(2n+1)+(2n+3)+(2n+5)+\ldots+(4n-1)}_{\hookrightarrow = 3n^2}+(n+1) = 3n^2+(n+1)$$
$$+n+1 = 3n^2+n+1$$
$$3n^2+n+1 = 3n^2+n+1 \checkmark$$

□

1.12

For $n \in \mathbb{N}$, let $n!$ denote the product $1\cdot 2\cdot 3\cdots n$. Also let $0!=1$ and define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{for } k=0,1,\ldots,n.$$

The binomial theorem asserts that

$$(a+b)^n = \binom{n}{0}a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\ldots+\binom{n}{n-1}ab^{n-1}+\binom{n}{n}b^n$$
$$= a^n+na^{n-1}b+\tfrac{1}{2}n(n-1)a^{n-2}b^2+\ldots+nab^{n-1}+b^n$$

(a) Verify the binomial theorem for $n=1,2$, and 3.

- $n=1$:
$$(a+b) = a^1+(1)a^{1-1}b \Rightarrow a+b = a+b$$
- $n=2$:
$$(a+b)^2 = a^2+2ab+\tfrac{1}{2}(2)(2-1)a^0b^2$$
$$a^2+2ab+b^2 = a^2+2ab+b^2$$
- $n=3$:
$$(a+b)^3 = \binom{3}{0}a^3+\binom{3}{1}a^2b+\binom{3}{2}ab^2+\binom{3}{3}b^3$$
$$a^3+3a^2b+3ab^2+b^3 = a^3+3a^2b+3ab^2+b^3 \; *$$

*
$$\binom{3}{0}=1 \quad \binom{3}{3}=1$$
$$\binom{3}{1}=\frac{3!}{1!(3-1)!}=\frac{3!}{2!}=3$$
$$\binom{3}{2}=\frac{3!}{2!(3-2)!}=\frac{3!}{2!}=3$$

(b) Show $\binom{n}{k}+\binom{n}{k-1}=\binom{n+1}{k}$ for $k=1,2,\dots,n$.

- $k=n$:

$$=\binom{n}{n}+\binom{n}{n-1}=\binom{n+1}{n}$$

$$=1+n=n+1$$

1 A number n choose the number below it, that is, $n-1$, is equal to n

AKA: $\frac{n!}{(n-1)!\,(n-(n-1))!}=\frac{n!}{(n-1)!}=\frac{n(n-1)!}{(n-1)!}=n$ □

2 similar logic

(c) Prove the binomial theorem using mathematical induction and part (b)

- The n^{th} proposition P_n is

$$(a+b)^n=\binom{n}{0}a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\dots+\binom{n}{n-1}ab^{n-1}+\binom{n}{n}b^n$$

- Base Case $n=1$:

$$(a+b)=\binom{1}{0}a+\binom{1}{1}a^0b$$

$$(a+b)=a+b \quad ✓$$

- Induction Step:

Suppose $(a+b)^n=\binom{n}{0}a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\dots+\binom{n}{n-1}ab^{n-1}+\binom{n}{n}b^n$ true

Want to prove P_{n+1} true

$$(a+b)^{n+1}=\binom{n+1}{0}a^{n+1}+\binom{n+1}{1}a^nb+\binom{n+1}{2}a^{n-1}b^2+\dots+\binom{n+1}{n}ab^n+\binom{n+1}{n+1}b^{n+1}$$

$$(a+b)^n(a+b)=$$

$$\binom{n}{0}a^{n+1}+\binom{n}{1}a^nb+\binom{n}{2}a^{n-1}b^2+\dots+\binom{n}{n-1}a^2b^{n-1}+\binom{n}{n}ab^n$$

$$+$$

$$\binom{n}{0}a^nb+\binom{n}{1}a^{n-1}b^2+\binom{n}{2}a^{n-2}b^3+\dots+\binom{n}{n-1}ab^n+\binom{n}{n}b^{n+1}=$$

$$\Rightarrow\binom{n}{0}a^{n+1}+a^nb\left[\binom{n}{1}+\binom{n}{0}\right]+a^{n-1}b^2\left[\binom{n}{2}+\binom{n}{1}\right]+\dots+ab^n\left(\binom{n}{n}+\binom{n}{n-1}\right)+\binom{n}{n}b^{n+1}$$

$$\Rightarrow\binom{n}{0}a^{n+1}+a^nb\left[\binom{n+1}{n}\right]^3+a^{n-1}b^2\left[\binom{n+1}{n}\right]^3+\dots+ab^n\binom{n+1}{n}^3+\binom{n}{n}b^{n+1}$$

$$\Rightarrow\underbrace{\binom{n}{0}}_{1}a^{n+1}+\binom{n+1}{n}a^nb+\binom{n+1}{n}a^{n-1}b^2+\dots+\binom{n+1}{n}ab^n+\underbrace{\binom{n}{n}}_{1}b^{n+1}=$$

$$\underbrace{\binom{n+1}{0}}_{1}a^{n+1}+\binom{n+1}{1}a^nb+\binom{n+1}{2}a^{n-1}b^2+\dots+\binom{n+1}{n}ab^n+\underbrace{\binom{n+1}{n+1}}_{1}b^{n+1}$$

□

2.1

Show $\sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{24}$, and $\sqrt{31}$ are not rational numbers.

- $\sqrt{3}$: x^2-3, $c_0=-3$, using 2.3 Corollary on page 10 of Ross, the only possible rational solutions are ± 1 and ± 3. We know $\sqrt{3}$ is a solution. Let's try our possible solutions: $(-1)^2-3 \neq 0$, $(1)^2-3 \neq 0$, $(-3)^2-3 \neq 0$, $(3)^2-3 \neq 0$. None of these possible rational solutions are solutions. Because $\sqrt{3}$ is a solution, it cannot be rational. ■

- $\sqrt{5}$: $x^2-5=0$, $c_0=-5$, only possible rational solutions are ± 1 and ± 5. $(-1)^2-5 \neq 0$, $(1)^2-5 \neq 0$, $(-5)^2-5 \neq 0$, $(5)^2-5 \neq 0$. None of these possible rational solutions are solutions; because $\sqrt{5}$ is a solution, it cannot be rational. ■

- $\sqrt{7}$: $x^2-7=0$, $c_0=-7$, only possible rational solutions are ± 1 and ± 7. $(-1)^2-7 \neq 0$, $(1)^2-7 \neq 0$, $(-7)^2-7 \neq 0$, $(7)^2-7 \neq 0$. None of these possible rational solutions are solutions; because $\sqrt{7}$ is a solution, it cannot be rational. ■

- $\sqrt{24}$: $x^2-24=0$, $c_0=-24$, only possible rational solutions are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12$, and ± 24! $\rightarrow$ $(-1)^2-24 \neq 0$, $(1)^2-24 \neq 0$, $(-2)^2-24 \neq 0$, $(2)^2-24 \neq 0$, $(-3)^2-24 \neq 0$, $(3)^2-24 \neq 0$, $(-4)^2-24 \neq 0$, $(4)^2-24 \neq 0$, $(-6)^2-24 \neq 0$, $(6)^2-24 \neq 0$, $(-8)^2-24 \neq 0$, $(8)^2-24 \neq 0$, $(-12)^2-24 \neq 0$, $(12)^2-24 \neq 0$, $(-24)^2-24 \neq 0$, $(24)^2-24 \neq 0$. None of these possible rational solutions are solutions; because $\sqrt{24}$ is a solution, it cannot be rational. ■

- $\sqrt{31}$: $x^2-31=0$, $c_0=-31$, only possible rational solutions are ± 1 and ± 31. $(-1)^2-31 \neq 0$, $(1)^2-31 \neq 0$, $(-31)^2-31 \neq 0$, $(31)^2-31 \neq 0$. None of these possible rational solutions are solutions; because $\sqrt{31}$ is a solution, it cannot be rational. ■

$\boxed{2.2}$ Show $\sqrt[3]{2}$, $\sqrt[7]{5}$ and $\sqrt[4]{13}$ are not rational numbers.

- $\sqrt[3]{2}$: $x^3-2=0$, $c_0=-2$ only possible rational solutions are ± 1 and ± 2. $(-1)^3-2\neq 0$, $(1)^3-2\neq 0$, $(-2)^3-2\neq 0$, $(2)^3-2\neq 0$. None of the possible solutions are solutions. Because $\sqrt[3]{2}$ is a solution, it cannot be rational. $\square$

- $\sqrt[7]{5}$: $x^7-5=0$, $c_0=-5$, only possible rational solutions are ± 1 and ± 5. $(-1)^7-5\neq 0$, $(1)^7-5\neq 0$, $(-5)^7-5\neq 0$, $(5)^7-5\neq 0$. None of the possible rational solutions are solutions. Because $\sqrt[7]{5}$ is a solution, it cannot be rational. $\square$

- $\sqrt[4]{13}$: $x^4-13=0$, $C_0=-13$, only possible rational solutions are ± 1 & ± 13. $(-1)^4-13\neq 0$, $(1)^4-13\neq 0$, $(-13)^4-13\neq 0$, $(13)^4-13\neq 0$. None of the possible rational solutions are solutions. Because $\sqrt[4]{13}$ is a solution, it cannot be rational. $\square$

$\boxed{2.7}$ Show the following irrational looking are actually rational numbers: (a) $\sqrt{4+2\sqrt{3}}-\sqrt{3}$ and (b) $\sqrt{6+4\sqrt{2}}-\sqrt{2}$

- (a) $\sqrt{4+2\sqrt{3}}-\sqrt{3}$

  $b=\sqrt{4+2\sqrt{3}}-\sqrt{3} \Rightarrow b+\sqrt{3}=\sqrt{4+2\sqrt{3}} \Rightarrow (b+\sqrt{3})^2=4+2\sqrt{3}$

  $\Rightarrow b^2+2\sqrt{3}b+3=4+2\sqrt{3} \Rightarrow b^2+2\sqrt{3}b+2\sqrt{3}-1=0$, which shows $b=\sqrt{4+2\sqrt{3}}-\sqrt{3}$ satisfies the polynomial eqn $x^2+2\sqrt{3}x+2\sqrt{3}-1=0$. $\square$

- (b) $\sqrt{6+4\sqrt{2}}-\sqrt{2}$

  $a=\sqrt{6+4\sqrt{2}}-\sqrt{2} \Rightarrow a+\sqrt{2}=\sqrt{6+4\sqrt{2}} \Rightarrow (a+\sqrt{2})^2=6+4\sqrt{2} \Rightarrow a^2+2\sqrt{2}a+2=6+4\sqrt{2}$

  $\Rightarrow a^2+2\sqrt{2}a-4+4\sqrt{2}=0$, which shows $a=\sqrt{6+4\sqrt{2}}-\sqrt{2}$ satisfies the polynomial equation $x^2+2\sqrt{2}x-4+4\sqrt{2}=0$. $\square$

$\boxed{4.11}$ Consider $a, b \in \mathbb{R}$ where $a<b$. Use Denseness of $\mathbb{Q}$ 4.7 to show there are infinitely many rationals between a and b.

- 4.7 Denseness of $\mathbb{Q}$:
  If $a, b \in \mathbb{R}$ and $a<b$, then there is a rational $r \in \mathbb{Q}$ such that $a<r_1<b$.
- Then there must also be another number that is in between a and r_1: $a<r_2<r_1$, and another between a and r_2: $a<r_3<r_2$.
- This pattern is repeated for r_n $n=1,2...$
- I'm not sure how to formalize this argument in proper mathematical language

$\boxed{4.14}$ Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let $A+B$ be the set of all sums $a+b$ where $a \in A$ and $b \in B$.

a) Prove $\sup(A+B) = \sup A + \sup B$. Hint: To show ~~sup A~~ $\sup A + \sup B \leq (\sup A + \sup B)$, show that for each $b \in B$, $\sup(A+B) - b$ is an upper bound for A, hence $\sup A \leq \sup(A+B) - b$. Then show $\sup(A+B) - \sup A$ is an upper bound for B.

- A is bounded $\Rightarrow A \subseteq [a_L, a_U]$ where a_L is the lower bound for A and a_u is the upper bound for A.
- B is bounded $\Rightarrow B \subseteq [b_L, b_u]$ where b_L is the lower bound for B and B_u is the upper bound for B.
- By the 4.4 Completeness Axiom on ross page 23, sup A and sup B exist and are real numbers $\Rightarrow \sup A + \sup B$ exists
- Not sure how to show for each $b \in B$, $A \leq \sup(A+B) - b$

7.5

(a) limit s_n where $s_n = \sqrt{n^2+1} - n \Rightarrow \lim \sqrt{n^2+1} - n$

$n=0$, $\lim s_n = 1$ / $n=2$, $\lim s_n = \sqrt{5}-2$ / $n=10$, $\lim s_n = \sqrt{101}-10$
$\sim .23$ ~ 0.049

$\Rightarrow$ limit $s_n = 0$

(b) $a = \lim(\sqrt{n^2+n} - n)$ $n=1$, $a=\sqrt{2}-1$ / $n=10$, $a=\sqrt{110}-10$ / $n=100$, $a=\sqrt{10100}-100$
$\sim .41$ $\sim .48$ $\sim .49$

$\Rightarrow \lim(\sqrt{n^2+n}-n) = \frac{1}{2}$

(c) $\lim(\sqrt{4n^2+n} - 2n)$ $\quad \sqrt{4n^2+n} - 2n \cdot \dfrac{\sqrt{4n^2+n}+2n}{\sqrt{4n^2+n}+2n}$

$= \lim \dfrac{4n^2+n-4n^2}{\sqrt{4n^2+n}+2n} = \lim \dfrac{n}{\sqrt{4n^2+n}+2n} = \dfrac{1}{4}$