Homework 1

1. 1.10

we need to prove $(2n+1)+(2n+3)+(2n+5)+\cdots(4n-1)=3n^2$

we can use proof by induction.

the base case would be
$n=1$, so $(2(1)+1) = 3(1)^2$
$3 = 3.$
so the base case is true.

Now we can check the inductive step.
we can assume that n is true and we want to show that it is true for n+1.

So we know $(2n+1)+(2n+3)+\cdots+(4n-1) = 3n^2.$
Now with $(n+1)$, we have.

$(2n+3)+(2n+5)+\cdots\cdots+(2(n+1)+(2(n+1)-1)) = 3(n+1)^2$
$= (2n+3)+(2n+5)+\cdots\cdots+(2n+2+2n+2-1)) = 3(n+1)^2$

$= (2n+3)+(2n+5)+\cdots+(4n+1)+(4n+3) = 3(n+1)^2$

Now we need to see if RHS = LHS.

RHS = $3(n+1)^2 = 3(n^2+2n+1) = 3n^2+6n+3.$

LHS = $(2n+3)+(2n+5)\cdots\cdots+(4n-1)+(4n+1)+(4n+3)$

$= 3n^2-(2n+1)+(4n+1)+(4n+3)$

$= 3n^2+6n+3$.

so here RHS = LHS.

so we see that n+1 satisfies the equation. so we have the inductive step is true.

so we have shown that

$(2n+1)+(2n+3)+(2n+5)+\cdots\cdots+(4n-1) = 3n^2$

by induction.

Answer = we have proven this by induction.

2. 1.12.

(a) we need to check the Binomial Theorem for $n=1,2,3.$

For $n=1$, we have

$(a+b)^1 = a+b$

Now we can check the formula.

so we have $\binom{1}{0} a^n + \binom{1}{1} a^0 b^1$

$\binom{1}{0} = \frac{1!}{0! \cdot 1!} = 1$

$\binom{1}{1} = \frac{1!}{1! \cdot 0!} = 1$

so $a^1 + b^1 = a + b.$

Now we check for $n = 2.$

$(a+b)^2 = a^2 + 2ab + b^2.$

The binomial formula is

$\binom{2}{0} \cdot a^2 + \binom{2}{1} a^1 \cdot b^1 + \binom{2}{2} a^{2-2} b^2.$

$\binom{2}{0} = \frac{2!}{0! \cdot 2!} = \frac{2!}{2!} = 1$

$\binom{2}{1} = \frac{2!}{1! \cdot 1!} = \frac{2}{1} = 2.$

$\binom{2}{2} = \frac{2!}{2! \cdot 0!} = \frac{2!}{2!} = 1.$

$= 1 \cdot a^2 + 2 \cdot a \cdot b + 1 \cdot a^0 b^2$

$= a^2 + 2ab + b^2.$

we can see that they are equal.

Now we need to check for $n = 3.$

$$(a+b)^3 = (a+b)(a^2+2ab+b^2)$$
$$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$$
$$= a^3 + 3a^2b + 3b^2 + b^3.$$

The binomial formula is.

$$\binom{3}{0} \cdot a^3 + \binom{3}{1} a^{3-1} \cdot b^1 + \binom{3}{2} a^{3-2} \cdot b^2$$
$$+ \binom{3}{3} \cdot a^{3-3} \cdot b^3.$$

$$\binom{3}{0} = 1$$

$$\binom{3}{1} = \frac{3!}{1! \cdot 2!} = \frac{3!}{2!} = 3.$$

$$\binom{3}{2} = \frac{3!}{2! \cdot 1!} = \frac{3!}{2!} = 3.$$

$$\binom{3}{3} = 1.$$

So we get the equation as

$$1 \cdot a^3 + 3 \cdot a^2b + 3ab^2 + 1 \cdot b^3$$
$$= a^3 + 3a^2b + 3ab^2 + b^3.$$

We can see that this is the same.
So we have shown for $n=1,2,3$.

Answer = we have verified the binomial theorem for $n = 1, 2, 3$.

(b). We need to show that

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

We know that

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

$$\binom{n}{k-1} = \frac{n!}{(k-1)! \cdot (n-(k-1))!}$$

$$\frac{n!}{k! \cdot (n-k)!} + \frac{n!}{(k-1)! \cdot (n-k+1)!}$$

$$\frac{n!}{k \cdot (k-1)! \cdot (n-k)!} + \frac{n!}{(k-1)! \cdot (n-k)! \cdot (n-k+1)}$$

$$\frac{n! \cdot (n-k+1) + n! \cdot k}{k \cdot (k-1)! \cdot (n-k)! \cdot (n-k+1)}$$

$$= \frac{n! \cdot (n-k+k+1)}{k! \cdot (n-k)! \cdot (n-k+1)}$$

$$= \frac{(n+1)!}{k! \cdot (n+1-k)!}$$

$$= \frac{(n+1)!}{k! \cdot ((n+1)-k)!} = \binom{n+1}{k}.$$

so here we have $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

Answer: we have shown that $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

(C) we need to prove the binomial theorem.

we need to prove it using mathematical induction.
so the basis step is when $n=1$.

$(a+b)^1 = a+b$

$\binom{n}{0} a + \binom{n}{1} b = 1 \cdot a + 1 \cdot b = a+b$

so it is true for the basis step

Now, we need the inductive step. Here we assume that it is true for n terms and we want to show that it is true for $(n+1)$ terms so we know that

$(a+b)^n = \binom{n}{0}a^n + \cdots\cdots\cdots + \binom{n}{n}a^0 b^n$

$(a+b)^{n+1} = (a+b)^n \cdot (a+b)$

$(a+b)\cdot\left(\binom{n}{0}a^n + \cdots\cdots + \binom{n}{n}a^0 b^n\right)$

$= \binom{n}{0}a^{n+1} + \binom{n}{0}a^n b + \binom{n}{1}a^n b + \binom{n}{1}a^{n-1}b^2 \cdots\cdots$

$+ \binom{n}{n}ab^n + \binom{n}{n}a^0 b^{n+1}.$

so we get:

$\binom{n}{0}a^{n+1} + \left[\binom{n}{1}+\binom{n}{0}\right]a^n b + \cdots + \left[\binom{n}{n}+\binom{n}{n-1}\right]ab^n$

$+ \binom{n}{n} b^{n+1}$

from part (b), we know that $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

so we get $\binom{n}{0}a^{n+1} + \binom{n+1}{1}a^n b + \binom{n+1}{2}a^{n-1}b^2$

$+ \binom{n+1}{n}ab^n + \binom{n}{n}b^n$

Also we know that $\binom{n}{0} = 1 = \binom{n+1}{0}$.

$\binom{n}{n} = 1 = \binom{n+1}{n+1}$.

so we can replace this to get.

$(a+b)^n = \binom{n+1}{0} a^{n+1} + \cdots\cdots + \binom{n+1}{n+1} b^{n+1}.$

so now we see that this is true when n is true.

So we can say that we have proven this using mathematical induction.

Answer > We have proven this statement using induction.

3. 2.1

we need to show that $\sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{24}$ and $\sqrt{31}$ are not rational numbers.

Here $\sqrt{3}$ can be checked first

$x = \sqrt{3}$, $x^2 - 3 = 0$

we know that are rational solution of this equation must be an integer that divides 3

so the integers are ± 1 and ± 3.

so we can check if these values work.

$(1)^2 - 3 = -2 \neq 0$ $(-1)^2 - 3 = -2 \neq 0$

$3^2 - 3 = 6 \neq 0$, $(-3)^2 - 3 = 6 \neq 0$.

So there are no solutions that are rational.

now $\sqrt{5}$.

$x^2 - 5 = 0$

The possible values are $\pm 1, \pm 5$.

we can check these values

$(1)^2 - 5 = -4$, $(-1)^2 - 5 = -4$.

$5^2 - 5 = 20$, $(-5)^2 - 5 = 20$

None of these equal 0.

so $\sqrt{5}$ is irrational.

$\sqrt{7}$ would be

$x^2 - 7 = 0$.

Possible values are $\pm 1, \pm 7$.

$1, -1$ would give us -6

$7, -7$, would give us 42.

None of these equal 0, so there is no rational solution, so $\sqrt{7}$ is irrational.

$\sqrt{24}$ would be.

$x^2 - 24 = 0$.

Possible values are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6$
$\pm 8, \pm 12, \pm 24$.

For ± 1 we get -23

For ± 2 we get -20

$\pm 3 \to -15$

$\pm 4 \to -10$

$\pm 6 \to 12$

$\pm 8 \rightarrow 40.$

$\pm 12 \rightarrow 120$

$\pm 24 \rightarrow 24 \cdot 23$

None of these equal 0, so there is no rational solution.

so $\sqrt{24}$ is irrational.

$\sqrt{31}$ would be.

Possible values are $\pm 1, \pm 31.$

$\pm 1 \rightarrow \quad x^2 - 31 = 0$

$= -30.$

$\pm 31 = 31 \cdot 30$

$= 930.$

None of these equal 0, so there is no rational solution, so $\sqrt{31}$ is irrational.

we have shown that these are not rational numbers

Answer = we have shown that $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{24}$ and $\sqrt{31}$ are not rational numbers.

4. 2.2

we need to show that $\sqrt[3]{2}$, $\sqrt[7]{5}$ and $\sqrt[4]{13}$ are not rational numbers.

For $\sqrt[3]{2}$, we have,

$$x = \sqrt[3]{2}$$

$$x^3 = 2$$

$$x^3 - 2 = 0.$$

Possible values are $\pm 1, \pm 2$.

$1 \to -1$

$-1 \to -3$

$2 \to 6$

$-2 \to -10$.

There is no rational solution, so $\sqrt[3]{2}$ is not rational.

For $\sqrt[7]{5}$, we have

$$x^7 - 5 = 0$$

Possible values are $\pm 1, \pm 5$.

$1 \to -4$

$-1 \to -6$

$5 \to 5^7 - 5 \neq 0.$

$-5 \to (-5)^7 - 5 = -5^7 - 5 \neq 0.$

so there is no rational solution, so $\sqrt[7]{5}$ is irrational.

For $\sqrt[4]{13}$, we have.

$x^4 - 13 = 0.$

Possible values are $\pm 1, \pm 13.$

$\pm 1 \rightarrow -12 \neq 0$

$\pm 13 \rightarrow (13)^4 - 13 \neq 0.$

So there is no rational solution, so.

$\sqrt[4]{13}$ is irrational.

So we have shown that there are not rational numbers.

Answer = we have shown that $\sqrt[3]{2}$, $\sqrt[7]{5}$ and $\sqrt[4]{13}$ are not rational numbers.

5. 2.7.

(a) $\sqrt{4+2\sqrt{3}} - \sqrt{3}.$

$$x = \sqrt{4+2\sqrt{3}} - \sqrt{3}.$$

$$x + \sqrt{3} = \sqrt{4+2\sqrt{3}}.$$

$$(x+\sqrt{3})^2 = \left(\sqrt{4+2\sqrt{3}}\right)^2.$$

$$x^2 + 3 + 2x\sqrt{3} = 4 + 2\sqrt{3}$$

$$x^2 + 3 + 2x\sqrt{3} - 4 = 2\sqrt{3}$$

$$x^2 - 1 + 2x\sqrt{3} = 2\sqrt{3}.$$

$$\frac{1}{2}\left(x^2 - 1 + 2\sqrt{3}x\right) = \sqrt{3}$$

$$\frac{1}{4}\left(x^4 + 10x^2 + 4\sqrt{3}x^3 - 4\sqrt{3}x + 1\right) = 3.$$

$$x^4 + 10x^2 + 4\sqrt{3}x^3 - 4\sqrt{3}x - 11 = 0.$$

Here we can see that $\pm 1, \pm 11$ are possible integer values.

when we have. 1 and -1 we see that the equation equals 0.

So we can get rational solutions as. 1 and -1

So here we get a rational root.

So this would mean that this is a rational number.

Answer: we have shown that it is a rational number.

(b) $\sqrt{6 + 4\sqrt{2}} - \sqrt{2}.$

$$6 + 4\sqrt{2} = (2 + \sqrt{2})^2$$

so we have that

$$\sqrt{6 + 4\sqrt{2}} = \sqrt{(2 + \sqrt{2})^2}$$

so we get $(2+\sqrt{2})$

$$2+\sqrt{2} - \sqrt{2} = 2$$

Here 2 is a rational number
so we see that this is a rational number.

Answer: we have shown that this is a rational number.

6. 3.6

(a) $|a+b+c| \leq |a| + |b| + |c|$.

we can say that $a+b = d$.

$|a+b+c| = |d+c|$

By the triangle inequality, we know that.

$|d+c| \leq |d| + |c|$

$|d| + |c| = |a+b| + |c|$

$|a+b| \leq |a| + |b|$

$|a+b| + |c| \leq |a| + |b| + |c|.$

so we see that it follows that

$|a+b+c| \leq |a| + |b| + |c|.$

Answer = we have shown that $|a+b+c| \leq |a| + |b| + |c|$

(b). Now we need to prove that
$|a_1 + \cdots\cdots + a_n| \leq |a_1| + \cdots\cdots + |a_n|.$

so here the base case is when
$n=1$
$|a_1| \leq |a_1|$
we see that this is true as $|a_1| = |a_1|$
Also when $n=2$, we see that
$|a_1 + a_2| \leq |a_1| + |a_2|$ based on the triangle inequality theorem.

Now we need to check the inductive step
if we assume it is true for n, we need to show it is true for $n+1$.

so as we know that it is true for n, we know that

$|a_1 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$.

now if we have n+1, we have.

$|a_1 + a_2 + \cdots + a_n + a_{n+1}|$

Here we can say that

$x = a_1 + a_2 + \cdots + a_{n-1} + a_n$.

so we get

$|x + a_{n+1}|$

Based on the triangle inequality property, we know that

$|x + a_{n+1}| \leq |x| + |a_{n+1}|$

$|x| = |a_1 + \cdots + a_n|$

Here we know that

$|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|$.

so this would mean that

$|x + a_{n+1}| \leq |x| + |a_{n+1}| \leq |a_1| + |a_2| + \cdots + |a_n| + |a_{n+1}|$.

so we see that $|a_1+\cdots\cdots+a_{n+1}| \leq |a_1|+\cdots\cdots+|a_{n+1}|$.

so we see that it is true for $n+1$.

so we can say that it is true for any n if $n-1$ is true.

so we have proved it using induction.

Answer = we have proven that
$|a_1+a_2+\cdots+a_{n-1}+a_n| \leq |a_1|+|a_2|+\cdots+|a_{n-1}|+|a_n|$

7: 4.11

we need to show that there are infinitely many rationals between a and b.

Based on Density, we know that
If $a, b \in \mathbb{R}$ and $a<b$ then there is a rational $r \in \mathbb{Q}$ such that $a<r<b$

so based on what we know, we can say that we have some r such that $a<r<b$

Here as $\mathbb{Q} \subset \mathbb{R}$, we know that $r \in \mathbb{R}$.

So we can use the denseness of $\mathbb{Q}$ to also show that $a < r_1 < r$ where $a < r < b$.

So we can do this process multiple times to see that

$a < r_n < \ldots\ldots < r < b$

So we can do a proof by induction to show that there are infinitely many rational numbers.

Base step would be, if we have $a < b$, then by the denseness of $\mathbb{Q}$, we know that $r \in \mathbb{Q}$, is such that $a < r < b$, so we have the base step.

For the inductive step, we assume that it is true for $n-1$ terms. So we need to show that it is true for n terms.

so we know that

$a < r_{n-1} < \dots\dots r < b.$

so here as $r_{n-1} \in \mathbb{Q}$ and $\mathbb{Q} \subset \mathbb{R}$, we have. $r_{n-1} \subset \mathbb{R}$.

so we can again use denseness of $\mathbb{Q}$ to say that we have.

$a < r_n < r_{n-1}$

so we have $a < r_n < r_{n-1} \dots\dots < r < b.$

So as we have this as true and as we have infinite number of natural and real numbers, we can say that there are infinite number of rational numbers between a and b.

Answer: we have proven the statement using the denseness of $\mathbb{Q}$.

Q. 4.14.

(a) $\sup(A+B) = \sup A + \sup B$

so here we have an equality. we can prove this by showing each one as being subsets of each other.

we can take

$c \in (A+B)$.

we know that $c \leq \sup(A+B)$ for all c.

so we can say that

$c = a + b$ for $a \in A$ and $b \in B$.

so as $C = a+b$

and $a \leq \sup(A)$ and $b \leq \sup(b)$.

we can say that

$c \leq \sup(A) + \sup(b)$.

this is true for $\forall c \in (A+B)$.

so we can say that

$\forall c \in (A+B)$, $c \leq \sup(A) + \sup(B)$.

so we can see that $\sup(A) + \sup(B)$ is an upper bound for $(A+B)$.

so we see that $\sup(A+B) \leq \sup(A) + \sup(B)$

Now we need to show the other direction where $\sup(A) + \sup(B) \le \sup(A+B)$

Here we know that
$a+b \in (A+B)$ and we see that

$$a+b \le \sup(A+B)$$

$$a \le \sup(A+B) - b.$$

so from this we can see that $\forall b$, $\sup(A+B)-b$ is an upper bound for A. so this means that

$$\sup(A) \le \sup(A+B) - b$$

now we can rearrange this to get

$$b \le \sup(A+B) - \sup(A)$$

so we see that this is an upper bound for B

so $$\sup(B) \le \sup(A+B) - \sup(A)$$

$$\sup(B) + \sup(A) \le \sup(A+B)$$

So we have shown that $\sup(A)+\sup(b) \leq \sup(A+B)$.

so from this we can see that

$$\sup(A+B) = \sup(A) + \sup(b).$$

Answer = we have shown that $\sup(A+B) = \sup(A) + \sup(B)$

(b). we need to show that

$$\inf(A+B) = \inf(A) + \inf(B).$$

So here we can take

$c \in (A+B)$,

we know that $c \geq \inf(A+B)$

Based on the definition of inf.

so we can say that

$c = a+b$, where $a \in A$ and $b \in B$.

Here we know that $a \geq \inf(A)$ and $b \geq \inf(B)$.

so $c = a+b \geq \inf(A) + \inf(B)$.

so $\quad c \geq \inf(A) + \inf(B)$

Here we can see that $\forall c$, we have $c \geq \inf(A) + \inf(B)$, so here $\inf(A) + \inf(B)$ is a lower bound.

So we can say that.

$$\inf(A+B) \geq \inf(A) + \inf(B)$$

We can now look at the other direction

Here as $\quad a+b \geq \inf(a+b)$, we know that

$$a \geq \inf(A+B) - b$$

so $\forall b$, we see that $\inf(A+B) - b$ is a lower bound.

so, $\quad \inf(A) \geq \inf(A+B) - b$

Now we have

$$b \geq \inf(A+B) - \inf(A).$$

Here we see that inf(A+B) − inf(A) is a lower bound for B.

so we have inf(B) ≥ inf(A+B) − inf(A)

So now as we have both directions, we can say that

$\inf(A+B) = \inf(A) + \inf(B)$

Answer = we have proven that $\inf(A+B) = \inf(A) + \inf(B)$

9. 7.5

(a) $\lim s_n$, $s_n = \sqrt{n^2+1} - n$.

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sqrt{n^2+1} - n.$$

we can multiply both by $\sqrt{n^2+1} + n$.

$$\left(\sqrt{n^2+1} - n\right) \cdot \frac{\left(\sqrt{n^2+1} + n\right)}{\left(\sqrt{n^2+1} + n\right)}$$

$$= \frac{(n^2+1) - n^2}{\sqrt{n^2+1} + n}$$

$$= \frac{1}{\sqrt{n^2+1}+n}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}+n} = 0.$$

so here $\lim_{n\to\infty} S_n = 0.$

Answer $= \lim_{n\to\infty} S_n = 0.$

(b). $\lim(\sqrt{n^2+n} - n).$

so here we have $\lim_{n\to\infty} \sqrt{n^2+n} - n.$

we can multiply by $\dfrac{\sqrt{n^2+n} - n}{\sqrt{n^2+n} - n}.$

we get

$$\lim_{n\to\infty} \frac{(n^2+n) - n^2}{\sqrt{n^2+n} - n} = \frac{n}{\sqrt{n^2+n} - n}.$$

Now we can divide the numerator and denominator by n.

we get,
$$\frac{1}{\sqrt{\frac{n^2+n}{n^2}}+1}$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{\frac{n^2+n}{n^2}}+1} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}.$$

$$\lim\left(\sqrt{n^2+n}-n\right) = \frac{1}{2}.$$

Answer $= \lim\left(\sqrt{n^2+n}-n\right) = \frac{1}{2}$.

(c) $\lim\left(\sqrt{4n^2+n}-2n\right)$

we can multiply by $\frac{\sqrt{4n^2+n}+2n}{\sqrt{4n^2+n}+2n}$.

$$= \lim_{n\to\infty}\frac{(4n^2+n)-4n^2}{\sqrt{4n^2+n}+2n}.$$

$$= \lim_{n \to \infty} \frac{n}{\sqrt{4n^2+n} + 2n}$$

we can divide by n.
we get

$$\lim_{n \to \infty} \frac{1}{\sqrt{\frac{4n^2+n}{n^2}} + 2}$$

$$\lim_{n \to \infty} \frac{1}{\sqrt{4+\frac{1}{n}} + 2}$$

Here we have.

$$\lim_{n \to \infty} \frac{1}{\sqrt{4+\frac{1}{n}} + 2} = \frac{1}{\sqrt{4+0} + 2} = \frac{1}{2+2}$$

$$= \frac{1}{4}.$$

$$\lim_{n \to \infty} \left(\sqrt{4n^2+n} - 2n\right) = \frac{1}{4}.$$

Answer $= \lim \left(\sqrt{4n^2+n} - 2n\right) = \frac{1}{4}$