Homework 10

1. 33.4.

Here we need to give an example of a function f on $[0,1]$ that is not integrable for which $|f|$ is integrable.

So we know f is not integrable when $U(f) \neq L(f)$.

So we need an example where $U(f) \neq L(f)$ but $U(|f|) = L(|f|)$.

So we can look at example 2 from §32.

Here we can modify such that

$f(x) = 1$ for rational x in $[a,b]$
$f(x) = -1$ for irrational in $[a,b]$.

So here we can find $U(f)$ and $L(f)$

$U(f,P)$, where
$P = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$

$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k])\cdot(t_k - t_{k-1}).$$

$$M(f,[t_{k-1},t_k]) = 1$$

so
$$U(f,P) = \sum_{k=1}^{n} 1\cdot(t_k - t_{k-1})$$
$$= (b-a) = 1-0 = 1.$$

$$L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k])\cdot(t_k - t_{k-1})$$

$$m(f,[t_{k-1},t_k]) = -1.$$

so
$$L(f,P) = \sum_{k=1}^{n} -1\,(t_k - t_{k-1})$$
$$= -1\cdot(1-0)$$
$$= -1.$$

so here $U(f) = 1$
$L(f) = -1$

so here we can see that $U(f) \neq L(f)$.
so f is not integrable.

now we can look at $|f|$.

$$U(|f|, P) = \sum_{k=1}^{n} M(|f|, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$$

$$= M(|f|, [t_{k-1}, t_k]) = 1.$$

$$= \sum 1 \cdot (t_k - t_{k-1})$$

$$= (1-0) = 1.$$

$$L(|f|, P) = \sum_{k=1}^{n} m(|f|, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$$

$$= m(|f|, [t_{k-1}, t_k]) = 1.$$

so $L(|f|, P) = \sum_{k=1}^{n} 1 \cdot (t_k - t_{k-1})$

$$= (1-0) = 1.$$

so $U(|f|) = 1.$

$L(|f|) = 1$

so $U(|f|) = L(|f|) = 1.$

so here we can see that $|f|$ is integrable.

so we have shown that f is not integrable and $|f|$ is integrable.

Answer = we have given an example for this question.

2. ~~33.7~~

(a). Here we need to show that

$$U(f^2, P) - L(f^2, P) \leq 2B[U(f, P) - L(f, P)].$$

Here we know that
f is bounded on $[a, b]$, and we have $B > 0$ such that $|f(x)| \leq B$.

so here we can say that

$$f(x)^2 - f(y)^2 = [f(x) + f(y)]\,[f(x) - f(y)].$$

so for $x, y \in [a, b]$, we know this is true.

So here we know that.

$|f(x)| \leq B$

so $[f(x) + f(y)] \leq 2B$.

so we get

$f(x)^2 - f(y)^2 \leq 2B[f(x) - f(y)]$.

now we need to convert $f(x)$ to $U(f, P)$ and $f(y)$ to $L(f, P)$.

so to convert this, we can say that

$S = (t_n, t_{n+1})$

so here we get that

$\sup\{f(x)^2 - f(y)^2\} \leq 2B \sup\{f(x) - f(y)\}$

for $x, y \in S$, we see that.

$\sup\{f(x)^2 - f(y)^2\} = \sup\{f(x)^2\} - \inf\{f(y)^2\}$

$\sup\{f(x)^2\} = M(f^2, S)$

$\inf\{f(y)^2\} = m(f^2, S)$

we also ~~get~~ get $\sup\{f(x) - f(y)\}$.
$$= M(f,S) - m(f,S).$$

so we have.

$$M(f^2,S) - m(f^2,S) \leq 2B(M(f,S) - m(f,S))$$

so from this we get that.

$$U(f^2,P) - L(f^2,P) \leq 2B(U(f,P) - L(f,P)).$$

So we have shown that

$$U(f^2,P) - L(f^2,P) \leq 2B(U(f,P) - L(f,P))$$

Answer = we have shown the statement.

(b). we need to show that if f is integrable then f^2 is integrable on $[a,b]$.

So if f is integrable, we know that.

$$U(f) = L(f).$$

so from part (a), we have.

$$U(f^2,P) - L(f^2,P) \leq 2B[U(f,P) - L(f,P)].$$

Here $U(f) = U(f,P)$

$L(f) = L(f,P)$.

so $U(f^2,P) - L(f^2,P) \leq 2B(0)$.

so $U(f^2,P) - L(f^2,P) \leq 0$.

we know that $U(f^2,P) \geq L(f^2,P)$.

so here we see that
$U(f^2,P) - L(f^2,P) \leq 0$, when

$$U(f^2,P) = L(f^2,P)$$

so $U(f^2) = L(f^2)$, so f^2 is integrable.

we can also see that
we know for $\text{mesh}(P) < \delta$, we have

$$U(L,P) - f(L,P) < \frac{\varepsilon}{2B}.$$

so here

so here

$U(f^2, \cancel{P}) - L(f^2, P) \leq 2B \cdot \left(\frac{\varepsilon}{2B} \right)$.

$U(f^2, P) - L(f^2, P) \leq \varepsilon$.

so for each $\varepsilon > 0$, $\exists \delta > 0$ such that for mesh$(P) < \delta$, we have that as $U(f^2, P) - L(f^2, P) \leq \varepsilon$, so it is integrable.

so we have that f^2 is integrable on $[a, b]$.

Answer: we have shown the statement.

3. 33.13.

So here we have $\int_a^b f = \int_a^b g$.

we need to show that $f(x) = g(x)$ for some x in (a, b).

By the intermediate value theorem for integrals, we know that if f is a continuous function on $[a, b]$, then for at least one x in (a, b), we have.

$$f(x) = \frac{1}{b-a} \int_a^b f.$$

so here we have some x for which this is true.

we know that $\int_a^b f = \int_a^b g.$

$$\text{so } f(x) = \frac{1}{b-a} \int_a^b g. = \frac{1}{b-a} \int_a^b f$$

so we can see that as we have the above statements, we can see that as f and g are continuous.
we have that $(f-g)$ is continuous.

$$\text{so } (f-g)(x) = \frac{1}{b-a} \left[\int_a^b f - \int_a^b g \right].$$

$$(f-g)(x) = \frac{1}{b-a} (0)$$

$$(f-g)(x) = f(x) - g(x) = 0.$$

so $f(x) = g(x)$ for some

so $f(x) - g(x)$ for some
x as we know that this is true because of the intermediate value theorem.

so for some $x \in (a, b)$, we get
$f(x) - g(x) = 0$, so $f(x) = g(x)$

we have gotten the statement.

Answer $\Rightarrow$ we have proven the statement

4. 35.4

(a) $\int_0^{\pi/2} x \, dF(x)$

so $F(t) = \sin t$

so $F'(t) = \cos(t)$

we know that $\cos(t)$ is continuous, so

we see that

$$\int_0^{\pi/2} x \, dF(x) = \int_0^{\pi/2} x \cdot F'(x) \, dx$$

$$\int_0^{\pi/2} x \cdot \cos(x) \, dx$$

we can use integration by Parts

$u = x$
$du = 1dx$
$dv = \cos(x)\,dx$
$v = \sin x.$

$$= x\sin x - \int \sin x.\,dx$$

$$x\sin x + \cos x \Big|_0^{\pi/2}$$

$$= \frac{\pi}{2} - 1.$$

So we get $\frac{\pi}{2} - 1.$

Answer $= \frac{\pi}{2} - 1.$

(b). $\int_{-\pi/2}^{\pi/2} x\,dF(x).$

Here we have $F(t) = \sin t,\ t \in [-\pi/2, \pi/2]$

$F'(t) = \cos(t).$

so as $\cos(t)$ is continuous, we have

$$\int_{-\pi/2}^{\pi/2} x\, dF(x) = \int_{-\pi/2}^{\pi/2} x F'(x)\, dx$$

$$= \int_{-\pi/2}^{\pi/2} x \cos(x)\, dx$$

$$= x\sin x + \cos x \Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{\pi}{2} - \left(\frac{\pi}{2}\right).$$

$$= 0$$

$$= 0$$

Answer $= 0$

5. 35.9.

(a). we need to show that

$$\int_a^b f\, dF = f(x)[F(b) - F(a)]$$ for some x in $[a, b]$.

so here we would have $f(x)$ and as it is continuous we know that f is bounded.

so that would mean that we have some $m, n \in [a,b]$ such that.

$f(m) \leq f(x) \leq f(n)$. From ~~this~~ we can get ~~that~~

$$\int_a^b f(m)\,dF \leq \int_a^b f(x)\,dF \leq \int_a^b f(n)\,dF$$

so here we get that.

$$\int_a^b f(m)\,dF = f(m)[F(b) - F(a)]$$

$$\int f(n)\,dF = f(n)[F(b) - F(a)].$$

so we get that

$$\int_a^b f(x)\,dF \leq f(n)[F(b) - F(a)]$$

$$\int_a^b f(x)\,dF \geq f(m)[F(b) - F(a)]$$

so $f(m) \leq \dfrac{\int f(x)\,dF}{F(b)-F(a)} \leq f(M)$.

Now as we have this equation, we can use the intermediate value theorem to get that.

we have some $x \in [a,b]$ such that

$$f(x) = \frac{\int_a^b f(x)\,dF}{[F(b)-F(a)]}$$

$$f(x) = \frac{\int_a^b f\,dF}{[F(b)-F(a))}$$

So here we would have that

$$f(x) = \frac{\int_a^b f\,dF}{(F(b)-F(a))}$$

So $\int_a^b f\,dF = f(x)\cdot[F(b)-F(a)]$.

so $\int_a^b f\,dF = f(x)\cdot[F(b)-F(a)]$ for some x in $[a,b]$

Answer :- we have proven the statement