Homework 2.

1. 9.9

(a) If $\lim s_n = +\infty$, $\lim t_n = +\infty$.

Here as $\lim s_n = +\infty$, we can say that we have some m such that $s_n > m$ for some $n > N_1$.
So now we can say $N = \max\{N_0, N_1\}$.

So we can see that if $n > N$, then we have

$$s_n > m.$$

Here we know $s_n \leq t_n$, so we get
$t_n \geq s_n > m$.

So we can see that for any $m > 0$, we have a number N such that
$t_n > m$ if $n > N$.

So from this we can say that $\lim t_n = +\infty$

Answer = we have shown that $\lim t_n = +\infty$.

(b). If $\lim t_n = -\infty$, then $\lim s_n = -\infty$.

So here as $\lim t_n = -\infty$, we can say that for some N_1, where $n > N_1$ we have $t_n < m$ for $n > N_1$, where m is a real number.

So now we take $N = \max\{N_0, N_1\}$.

Here we see that if $n > N$, we have that $t_n < m$. We also know that for $n > N$, we have $s_n \leq t_n$.

So we can combine this to see that

$$s_n \leq t_n < m.$$

So we can see that for any $m < 0$, we have some N such that $s_n \leq t_n < m$ or $s_n < m$. if $n > N$.

So we can see that $\lim s_n = -\infty$.

$\lim s_n = -\infty$.

Answer: we have shown that $\lim s_n = -\infty$.

(c). if $\lim s_n$ and $\lim t_n$ exist, then $\lim s_n \le \lim t_n$

so here we can say that
$s = \lim s_n$
$t = \lim t_n$.

we can show that $t_n - s_n \geq 0$, or
that $t - s \geq 0$.
So we can say $t_n - s_n = a_n$
$t - s = a$.

so now we have to show or check
that $a \geq 0$.

so we can try a proof by contradiction.
we can take $a < 0$.

so now we have $a < 0$, and
$\varepsilon > 0$ such that $a + \varepsilon < 0$.

so as $\lim a_n = a$, we have N_1 such that
$a - \varepsilon < a_n < a + \varepsilon$, for $\forall n > N_1$.

Now we can take $N = \max\{N_0, N_1\}$.

so we see that $a_n < a + \varepsilon < 0$ for $\forall n \geq N$

So here we get $a_n < 0$ for $\forall n \geq N$

so we see that the assumption that $a < 0$ is incorrect.

so this means that $a \geq 0$ and $\lim a_n = a$, so $\lim a_n \geq 0$.

$a_n = t_n - s_n$, so

$\lim(t_n - s_n) \geq 0$.

$\lim t_n - \lim s_n \geq 0$.

$\lim t_n \geq \lim s_n$.

So we get that $\lim t_n \geq \lim s_n$.

Answer: we have shown ~~that~~ $\lim s_n \leq \lim t_n$

2. 9.15.

We need to show that $\lim_{n\to\infty} \frac{a^n}{n!} = 0$.

$$= \lim_{n\to\infty} \frac{a^n}{n!}$$

Here we can say that $S_n = \frac{a^n}{n!}$

we want to use the statement

If $\lim \left| \frac{S_{n+1}}{S_n} \right| < 1$, then $\lim S_n = 0$.

Here $S_{n+1} = \frac{a^{n+1}}{(n+1)!}$

$$\frac{S_{n+1}}{S_n} = \frac{a^{n+1}}{(n+1)!} \times \frac{n!}{a^n} = \frac{a}{n+1}.$$

$$\lim_{n\to\infty} \left| \frac{a}{n+1} \right| = \left(\frac{a}{\infty+1} \right) = \frac{a}{\infty} = 0.$$

So here we see that

$$\lim \left| \frac{S_{n+1}}{S_n} \right| = 0$$

As $0 < 1$, we get that

$\lim \left| \frac{s_{n+1}}{s_n} \right| < 1$, so from this we see that $\lim_{n \to \infty} s_n = 0$.

so we get that $\lim_{n \to \infty} \frac{a^n}{n!} = 0$.

So we have $\lim_{n \to \infty} \frac{a^n}{n!} = 0$.

Answer = we have ~~shown that~~ $\lim_{n \to \infty} \frac{a^n}{n!} = 0$

3. 10.7

we need to show that there is a sequence (s_n) such that $\lim s_n = \sup S$.

so here we see that we can say. $\sup S = x$

Now we can take some function like $1/n$ to show that $\lim s_n = \sup S$.

So as we know $\frac{1}{n} > 0 \ \forall n \in \mathbb{N}$.

we see that $x - \frac{1}{n} \neq x \ \forall n \in \mathbb{N}$.

so here as $x - \frac{1}{n} < x$, we have some s_n such that.

$$x - \frac{1}{n} < s_n < x.$$

now as we know that this is true, we can use the squeeze theorem to show that

$$\lim_{n\to\infty}\left(x - \frac{1}{n}\right) = \lim_{n\to\infty} x - \lim_{n\to\infty}\frac{1}{n}$$

$$= x$$

$$\lim_{n\to\infty}(x) = x$$

so we get $x \leq \lim_{n\to\infty} s_n \leq x$

so by the squeeze theorem we see that.

$\lim_{n\to\infty} s_n = x$.

Here $x = \sup S$

So we have $\lim S_n = \sup S$

So we got $\lim S_n = \sup S$.

Answer: we have shown that $\lim S_n = \sup S$

4. 10.8.

Here we want to show that
$\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$ is an increasing sequence.

We know that (s_n) is an increasing sequence so that means that
$s_{n+1} \geq s_n \quad \forall n \in \mathbb{N}$.

So here we can see what (σ_n) is

$\sigma_1 = \frac{1}{1}(s_1) = s_1$

$\sigma_2 = \frac{1}{2}(s_1 + s_2)$

$\sigma_3 = \frac{1}{3}(s_1 + s_2 + s_3)$

$\vdots \qquad \vdots$

$$\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n).$$

We can first show that $\sigma_2 > \sigma_1$.

$\sigma_2 > \sigma_1$ because.

$s_2 > s_1$ as we have.

$$\frac{1}{2}(s_1 + s_2) > \frac{1}{2}(s_1 + s_1) = s_1$$

so $\frac{1}{2}(s_1 + s_2) > s_1 = \sigma_1$.

Now we can show for $\sigma_3 > \sigma_2$.

$$\sigma_3 = \frac{1}{3}(s_1 + s_2 + s_3) > \frac{1}{3}(s_1 + 2s_2)$$

we need to show that

$$\frac{1}{3}(s_1 + 2s_2) > \frac{1}{2}(s_1 + s_2).$$

$$\frac{1}{3}s_1 + \frac{2}{3}s_2 > \frac{s_1}{2} + \frac{s_2}{2}$$

$$\frac{2}{3}s_2 - \frac{s_2}{2} > \frac{s_1}{2} - \frac{s_1}{3}$$

$$\frac{1}{6}s_2 > \frac{1}{6}s_1$$

$s_2 > s_1$ which we know is

true, so we get

$$\frac{1}{3}\left(s_1+s_2+s_3\right) > \frac{1}{3}\left(s_1+2s_2\right) > \frac{1}{2}\left(s_1+s_2\right).$$

so $\sigma_3 > \sigma_2$.

we can see that this would also be true for σ_n as here we can carry this method further recursively and simplify to see that $\sigma_n > \sigma_{n-1}$ based on what was shown above.

we can see that

$$s_{n+1} \geq s_n \geq \frac{1}{n}\left(s_1+\cdots\cdots+s_n\right)$$

$$s_{n+1} \geq \frac{1}{n}\left(s_1+\cdots\cdots+s_n\right)$$

so we see that

$$s_{n+1} \geq \frac{n+1}{n}\left(s_1+\cdots+s_n\right)-\left(s_1+\cdots+s_n\right)$$

we can simplify this to get that.

$$\frac{1}{n+1}\left(s_1+\cdots\cdots+s_{n+1}\right) \geq \frac{1}{n}\left(s_1+\cdots\cdots+s_n\right)$$

$$\underbrace{\qquad\qquad}_{\sigma_{n+1}} \qquad \underbrace{\qquad\qquad}_{\sigma_n}$$

so $\sigma_{n+1} \geq \sigma_n$.

so this would mean that (σ_n) is an increasing sequence.

Answer = we have shown that (σ_n) is an increasing sequence

5. 10.9

(a) we need to find s_2, s_3, s_4.

$s_1 = 1$.

$s_2 = \frac{1}{2} \cdot 1^2 = \frac{1}{2}$

$s_3 = \frac{2}{3} \cdot \left(\frac{1}{2}\right)^2 = \frac{2}{3} \cdot \frac{1}{4} = \frac{2}{12} = \frac{1}{6}$

$s_4 = \frac{3}{4} \cdot \left(\frac{1}{6}\right)^2 = \frac{3}{4} \cdot \frac{1}{36} = \frac{1}{48}$.

So $s_2 = 1/2$, $s_3 = 1/6$, $s_4 = 1/48$.

Answer $= s_2 = 1/2$

$s_3 = 1/6$

$s_4 = 1/48$

(b). Here we needs to show that a limit exists.

so we can see that

$s_{n+1} = \frac{n}{n+1} \cdot (s_n)^2$, $s_1 = 1$ is the sequence.

we can see that this is a decreasing sequence as we can see that

$s_{n+1} \leq s_n$

we can prove that it is a monotonically decreasing sequence.

we have already seen the base step where $s_2 \leq s_1$.

Now we need to see for s_{n+1}.

we can assume that $s_n \leq s_{n-1}$.

so here $s_{n+1} = \frac{n}{n+1} \cdot (s_n)^2$.

we need to show that $\frac{n}{n+1} \cdot (s_n)^2 \leq s_n$.

Here $s_n \leq 1$, so $(s_n)^2 \leq 1$ and $\frac{n}{n+1} < 1$.

so we see that as $s_n \leq 1$, $s_n \geq (s_n)^2$ and as $\frac{n}{n+1} < 1$, we can see that

$$\frac{n}{n+1} \cdot (s_n)^2 \leq s_n.$$

so we get $s_{n+1} \leq s_n$

So as s_n is monotonically decreasing and as $s_1 > 0$ and $s_n > 0$ as the values are positive, we can see that it is bounded below by 0.

so as we have decreasing and bounded below, we know that the sequence converges

so it converges so the limit exists

Answer = we have shown that $\lim s_n$ exists

(c) we need to show that $\lim s_n = 0$.

$s_{n+1} = \left(\frac{n}{n+1}\right) \cdot (s_n)^2$ ~~and~~

Here we can say that $\lim s_n = s$.

We know that $\lim s_n < s_1$, so
$\lim s_n < 1$, so $s < 1$.

Also based on subsequences, we know that
$\lim s_{n+1}$ will converge to the same limit.

so $\lim s_{n+1} = s$.

so $\lim s_{n+1} = \lim \left(\frac{n}{n+1}\right) \cdot (s_n^2)$

$= \lim \left(\frac{n}{n+1}\right) \cdot \lim (s_n^2)$.

$= 1 \cdot \lim (s_n)^2$

$\lim (s_{n+1}) = 1 \cdot s^2$

$s = s^2$

~~so~~ as $s = s^2$, we can see that
$s^2 - s = 0$ $\quad s(1-s) = 0$.

$s = 0$ or $s = 1$

as we know $s < 1$, we get

$S = 0.$

So $\lim s_n = s = 0.$

So we have $\lim s_n = 0.$

Answer = we have shown that $\lim s_n = 0$.

6. 10.10

(a) we need s_2, s_3, s_4.

$$s_2 = \frac{1}{3}(1+1)$$
$$= \frac{2}{3}$$

$$s_3 = \frac{1}{3}\left(\frac{2}{3}+1\right)$$
$$= \frac{1}{3}\cdot\frac{5}{3} = \frac{5}{9}.$$

$$s_4 = \frac{1}{3}\left(\frac{5}{9}+1\right)$$
$$= \frac{1}{3}\cdot\left(\frac{14}{9}\right)$$

$$= \frac{14}{27}.$$

$S_2 = 2/3$, $S_3 = 5/9$, $S_4 = 14/27$.

Answer = $S_2 = 2/3$
= $S_3 = 5/9$
= $14/27$

(b) we need to show that $S_n > \frac{1}{2}$ for all n.

so for the base case, we see that
$S_1 = 1 > \frac{1}{2}$.

$S_2 = \frac{2}{3} > \frac{1}{2}$.

so this is true.

we now need to check the inductive step, we can assume it is true for n and we need to show that it is true for n+1.

So here $S_{n+1} = \frac{1}{3}(S_n + 1)$.

Here $S_n > \frac{1}{}$.

so $\frac{1}{3}(s_n+1) > \frac{1}{3}(\frac{1}{2}+1) = \frac{1}{3}(\frac{3}{2})$.

$\frac{1}{3}(s_n+1) > \frac{1}{3}\cdot(\frac{3}{2}) = \frac{1}{2}$.

so we get $\frac{1}{3}(s_n+1) > \frac{1}{2}$

so we see that $s_{n+1} > \frac{1}{2}$.

so this mean that this is true by induction. So we see that $s_n > \frac{1}{2}$ for $\forall n$ if $s_{n-1} > \frac{1}{2}$ based on induction.

so we $s_n > \frac{1}{2}$ for all n.

Answer = We have show that $s_n > \frac{1}{2}$ for all n

(c) we need to show that the limit exist and we need to find the limit

we can see that it is a decreasing sequence. we can prove this.

Here we know that $s_n > \frac{1}{2}$.

So we need to show that $s_{n+1} \leq s_n$.

So as $s_n > \frac{1}{2}$, we have

$2s_n > 1$, we can add s_n on both sides to get

$3s_n \geq 1 + s_n$.

$s_n \geq \frac{1}{3}(s_n + 1)$.

Here we know that $s_{n+1} = \frac{1}{3}(s_n + 1)$.

So $s_n \geq s_{n+1}$

So as $s_{n+1} \leq s_n$, we can see that it is a decreasing sequence.

Also we know that it is bounded below by $1/2$. So as this is a bounded monotonically decreasing sequence, it will converge and so the limit exists.

Now we need to find the limit.

$\lim s_{n+1} = \lim\left(\frac{1}{3}(s_n + 1)\right)$

$$= \frac{1}{3} \text{Lim} (s_{n+1})$$

we can say $\lim s_n = s$

so $\frac{1}{3} (\lim (s_n) + \lim (1))$

$$= \frac{1}{3} (s + 1)$$

As $s_{n+1} < s_n$, we see that s_{n+1} would be a subsequence of s_n. As it is a convergence sequence, we can see that the values will be convergent at the same limit or value.

So $\lim s_{n+1} = s$

$$s = \frac{1}{3} (s+1)$$

$$3s = s+1$$

$$2s = 1$$

$$s = \frac{1}{2}.$$

$$s = \frac{1}{2}$$

so $\lim s_n = \lim s_{n+1} = \frac{1}{2}.$

so $\lim s_n = \frac{1}{2}$.

Answer = The limit is $\lim s_n = \frac{1}{2}$.

7. 10.11.

(a) we need to show that $\lim t_n$ exists.

$$t_1 = 1, \quad t_{n+1} = \left[1 - \frac{1}{4n^2}\right] \cdot t_n$$

So we can see that the sequence looks decreasing. we can prove this by induction.

The Base Case, here we need to show that $t_2 \le t_1$.

$$t_2 = \left[1 - \frac{1}{4}\right] \cdot 1 = \frac{3}{4}.$$

Here $\frac{3}{4} \le 1$, so it is true.

we need to check the inductive step. Here we assume that $t_n \le t_{n-1}$ and we need to show that $t_{n+1} \le t_n$.

$$t_{n+1} = \left[1 - \frac{1}{4n^2}\right] \cdot t_n \quad , \text{here } n > 1$$

so as $n > 1$, we see that

$$1 - \frac{1}{4n^2} \leq 1$$

so $t_{n+1} \leq t_n$.

so we see that this is true. so now we have shown by induction that $t_{n+1} \leq t_n$ for $n \geq 1$.

Also we can see that $t_n > 0$ $\forall n$, so we see that it is bounded below by 0.

so as we have a bounded monotonic sequence, we know that it converges

so as we have a bounded monotonically decreasing sequence, we know it converges and so the limit exists.

so $\lim t_n$ exists.

Answer = we have shown that $\lim t_n$ exists.

(b). we need to find $\lim t_n$.

we can say that $\lim t_n = t$.

As t_{n+1} is a subsequence of t_n, and as the sequence converges, we know ~~that~~ $\lim t_{n+1} = \lim t_n$.

$$\lim t_{n+1} = \lim\left(t_n - \frac{t_n}{4n^2}\right).$$

$$\lim t_{n+1} = \lim(t_n) - \frac{\lim(t_n)}{\lim(4n^2)}$$

$$\lim t_{n+1} = t - \frac{t}{\infty}$$

$$= t - 0 = t.$$

we get $\lim t_{n+1} = t$

so we get $t = t$

we see that the limit does exist, but we would need to use a different approach as we have only circled back to $\lim t_{n+1} = t$ using the above approach

so by looking at the equation of

t_{n+1}, we can split it up.

we can use a technique of converting this into a wallis product.

After getting the wallis product, we can use the limit theorem to find $\lim t_n$.

we see that $\lim t_n = \frac{2}{\pi}$.

so we get that $\lim t_n = \frac{2}{\pi}$.

Answer = The limit is $\lim t_n = \frac{2}{\pi}$

8. Squeeze test

$a_n \le b_n \le c_n$ and $L = \lim a_n = \lim c_n$
we need to show that $\lim b_n = L$.
So here we can see that for any $\varepsilon > 0, \exists n$ such that $n > N_1$ we have.

$|a_n - L| < \varepsilon$. , where N_1 is corresponding to $\varepsilon > 0$.

so $-\varepsilon < a_n - L < \varepsilon$.

$-\varepsilon + L < a_n < \varepsilon + L$.

so we see that for every $\varepsilon > 0$, there exists N_1 such that, $n > N_1$

$-\varepsilon + L < a_n < \varepsilon + L$.

Now, we can see that
for any $\varepsilon > 0$, there exists N_2 such that for $n > N_2$, we have.

$|c_n - L| < \varepsilon$.

$-\varepsilon < c_n - L < \varepsilon$

$-\varepsilon + L < c_n < \varepsilon + L$

so we see that this is true for $\forall \varepsilon > 0$ for some N_2 where $n > N_2$.

Now we can say that

$N_0 = \max\{N_1, N_2\}$.

so for $n > N_0$, we know that

$-\varepsilon+L < c_n < \varepsilon+L$

$-\varepsilon+L < a_n < \varepsilon+L.$

so we can see that

$-\varepsilon_1+L < a_n \leq b_n \leq c_n < \varepsilon+L.$

so as we know that

$a_n \leq b_n \leq c_n \; \forall n$, we can say that when $n > N_0$, we have.

$-\varepsilon+L < a_n \leq b_n \leq c_n < \varepsilon+L$

$-\varepsilon+L < b_n < \varepsilon+L.$

we can subtract L.

$-\varepsilon < b_n - L < \varepsilon.$

$= \quad |b_n - L| < \varepsilon.$

so here we see that for every $\varepsilon > 0$, there is some N such that.

$n > N$, we have.

$|b_n - L| < \varepsilon.$

so. $\lim b_n = L.$

we have that $\lim b_n = L.$

Answer = We have shown that $\lim b_n = L$
