Homework 3.

1. 10-6

(a) we need to prove that (s_n) is a cauchy sequence

Here we have $|s_{n+1} - s_n| < 2^{-n}$ $\forall n \in \mathbb{N}$

we need to show $|s_m - s_n| < \varepsilon$.

so we can say that

$s_m - s_n = (s_m - s_{m-1}) + (s_{m-1} - s_{m-2}) + \cdots \cdots (s_{n+1} - s_n)$

so we can see that

$|s_m - s_n| = |s_m - s_{m-1} + s_{m-1} + \cdots - s_n|$

$\leq |s_m - s_{m-1}| + |s_{m-1} + s_{m-2}| + \cdots \cdots |s_{n+1} - s_n|$

$|s_m - s_{m-1}| < 2^{-(m-1)}$

so we get that.

$|s_m - s_{m-1}| + \cdots + |s_{n+1} - s_n| < 2^{-(m-1)} + \cdots \cdots + 2^{-(n)}$.

we can see that we have.

$2^{-(m-1)} + \cdots + 2^{-n} < 2^{-n+1}$.

so here we have.

$|s_m - s_n| < 2^{-n+1}$

we need $2^{-n+1} < \varepsilon$.

so here we have that.

$n > 1 - \log_2 \varepsilon$

so here we can see that

$\forall \varepsilon > 0, \exists N$ such that $2^{-N+1} < \varepsilon$.

So for $m, n > N$, we have.

$|s_m - S_n| < 2^{-(n+1)} < 2^{-N+1}$

Here $2^{-N+1} < \varepsilon$, so we get

$|s_m - s_n| < \varepsilon$

So as we have $|s_m - s_n| < \varepsilon$, we have that it is a Cauchy sequence.

Now as we know that (s_n) is Cauchy we ~~know~~ that the sequence is Convergent.

Answer = we have shown that (s_n) is Cauchy and Convergent.

(b) Here we need to check $|s_{n+1} - s_n| < \frac{1}{n}$.

so here we have to show
$|s_m - s_n| < \varepsilon$, for $m, n > N$.

So here we have
$|s_m - s_n| = |s_m - s_{m-1} + s_{m-1} - s_{m-2} + \cdots\cdots - s_n|$.
we can use the triangle inequality to get.

$|s_m - s_{m-1} + s_{m-1} + \cdots - s_n| < |s_m - s_{m-1}| + \cdots\cdots |s_{n+1} - s_n|$

Here $|s_{n+1} - s_n| < \frac{1}{n}$.

so we have
$|s_m - s_{m-1}| + \cdots |s_{n+1} - s_n| < \frac{1}{m-1} + \cdots\cdots + \frac{1}{n}$.

so here we have
$$\frac{1}{m-1} + \cdots + \frac{1}{n} < \frac{1}{n} + \cdots\cdots + \frac{1}{n} = \frac{(m-n)}{n}.$$

so here we have.

$$|s_m - s_n| < \left(\frac{m-n}{n}\right)$$

so here $\frac{m-n}{n} < \varepsilon$.

$m-n < \varepsilon n \rightarrow m < \varepsilon n + n$.

Here based on this condition, we can see that the sequence not bounded.

So (s_n) is not bonded, so it is not convergent.

Answer = (s_n) is not bonded and so it is not convergent.

2. 11.2

(a) we need to give an example of a monotone subsequence.
The monotone subsequence for $a_n = (-1)^n$ would be.
$a_n = (-1)^n$ for all even n.

For b_n, we would have the subsequence as $n > a$

For C_n, we have that C_n is a monotone sequence and every sequence is a subsequence of itself.
So the monotone subsequence would just be (C_n).

For d_n, we have that (d_n) is a monotonically decreasing sequence, so a monotone subsequence would be
(d_n).

Answer $= A_n = (A_{2n})$
$= B_n = (B_{n>0})$
$= C_n = (C_n)$
$= d_n = (d_n)$.

(b) we need the set of its subsequential limits

For a_n, we see that the set of possible values are
$\{-1, 1\}$.

So the set of subsequential limits are
$\{-1, 1\}$.

For b_n, we see that
$b_n = 1/n$
so here we see that
$\lim_{n\to\infty} b_n = \frac{1}{\infty} = 0.$

so $\lim b_n = 0$.

so as b_n has a limit, all subsequences would have the same limit, so we have.

$\{0, \ldots\ldots, 0\} = \{0\}.$

so it is $\{0\}$.

For c_n, we see that
$c_n = n^2$.

Here we know that n^2 diverges, so this would mean that the set of subsequential limits is

$= \{\infty\}.$

For d_n, we have that $d_n = \frac{6n+4}{7n-3}$

$$\lim_{n\to\infty} \frac{6n+4}{7n-3} = \frac{\frac{6n+4}{n}}{\frac{7n-3}{n}} = \frac{6+4/n}{7-3/n}$$

$$\frac{\lim 6+4/n}{\lim 7-3/n} = \frac{6}{7}.$$

So as d_n has a limit, we see that all subsequences have the same limit.

So the set of subsequential limits would be $\{\frac{6}{7}\}$.

Answer $=$ $A_n = \{-1, 1\}$

$= B_n = \{0\}$.

$= C_n = \{\infty\}$

$= d_n = \{6/7\}$.

(c) Here we need to find limsup and liminf

For a_n, we have that

$$\limsup a_n = \lim_{n\to\infty} \sup\{a_n : n > N\}.$$

Here we have $\lim_{n\to\infty} \sup\{1\}$.

$= 1.$

Here $\liminf a_n = -1.$

For b_n, we have that we have a sequential limit.

So this means that

$\limsup b_n = \lim b_n = \liminf b_n.$

$\lim b_n = 0.$

So $\limsup b_n = 0$

$\liminf b_n = 0$

For C_n, we have that it diverges.

So this means that $\limsup C_n = \infty.$

Here as the set has only one value or element, we have

$\liminf = \infty$

For d_n, we have only one value in the set of sequential limits and it converges to $6/7.$

So $\limsup d_n = 6/7.$

$\liminf a_n = 6/7$

Answer = a_n = $\limsup a_n = 1$

= $\liminf a_n = -1$

= b_n = $\limsup b_n = 0$

= $\liminf b_n = 0$

= c_n = $\limsup c_n = \infty$

= $\liminf c_n = \infty$

= d_n = $\limsup d_n = 6/7$

= $\liminf d_n = 6/7$

(d) Here we see that

(a_n) does not converge as it is alternating.

(b_n) converges to 0.

(c_n) diverges to ∞.

(d_n) converges to $6/7$.

(e) we need to see which sequences are bonded.

For (a_n), the set of elements are
$\{-1, 1\}$.

The bounds are -1 and 1, so it is
bounded.

So we can now look at (b_n).

Here, as $n=1$, we have $\frac{1}{1} = 1$.

So here as $n \to \infty$, $\frac{1}{n} \to 0$

So we are bounded $[0, 1]$.

For c_n, it is unbounded as $\lim c_n = \infty$.

For d_n, we have monotonically decreasing
so it is bounded, with $[\frac{5}{2}, 6/7]$.

3. 11.3

(a). we need to get a monotone subsequence.
$s_n = \cos(\frac{n\pi}{3})$.

Here we see that we repeat -1 every $n = (6k+3)$ times.

So we have $(s_n)_{n=6k+3,\, k \in \mathbb{N}}$.

For $t_n = \frac{3}{4n+1}$.

Here this is a monotonically decreasing sequence. So t_n is a subsequence of itself and so (t_n) is a monotone subsequence.

$u_n = \left(-\frac{1}{2}\right)^n$.

Here we can see that if n is even, then it is monotonically decreasing.

So $\left(-\frac{1}{2}\right)^{2n}$ is a monotone subsequence.

(u_{2n})

$v_n = (-1)^n + \frac{1}{n}$.

Here when n is even it is monotonic decreasing.

so (V_{2n}) is a monotone subsequence.

Answer = S_n = (S_{6n+3})

= t_n = (t_n)

= u_n = (u_{2n})

= v_n = (v_{2n})

(b) we need the set of subsequential limits.

Here $\cos\left(\frac{n\pi}{3}\right) = \left\{ \frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \ldots \right\}$.

So the set of subsequential limits are

$\left\{ \frac{1}{2}, \frac{-1}{2}, -1, 1 \right\}$.

For t_n, here we see that

$\lim_{n\to\infty} \frac{3}{4n+1} = \frac{3}{\infty} = 0.$

So as t_n converges to 0, all subsequences converge to 0.

So we have $\{0\}$

So we have $\{0\}$.

$u_n = \left(-\frac{1}{2}\right)^n$

Here when n is odd $(-1)^n \cdot \frac{1}{2^n}$ we have

$\frac{-1}{2^\infty} = 0$

when n is even $\frac{1}{2^\infty} = 0.$

So here the set is $\{0\}$.

For v_n, we have
$(-1)^n + \frac{1}{n}$.

$\lim_{n\to\infty} (-1)^n + \frac{1}{n} = -1 + \frac{1}{\infty} = -1$

or $1 + \frac{1}{\infty} = 1$

So here the subsequential limits are $\{-1, 1\}$

Answer = $S_n = \{1/2, -1/2, -1, 1\}$.
$= t_n = \{0\}$.
$= u_n = \{0\}$.
$= v_n = \{-1, 1\}$.

c(i) we need the lim sup and liminf.

For s_n, we have that.

$\limsup s_n =$ sup(subsequential set)

$= \sup \{-1/2, 1/2, -1, 1\}$

$= 1$

$\liminf = \inf \{-1/2, 1/2, -1, 1\}$

$= -1.$

For t_n, as we have only one element, we will have.

$\limsup t_n = \lim t_n = \liminf t_n$

$= 0.$

For u_n, we have one element, so.

$\limsup u_n = \lim u_n = \liminf u_n$

$= 0.$

For v_n, we have.

$\limsup_n v_n = \sup \{-1, 1\} = 1$

$\liminf_n v_n = \inf \{-1, 1\} = -1.$

Answer = s_n = $\limsup s_n = 1$

= $\liminf s_n = -1$

= t_n = $\limsup t_n = 0$

= $\liminf t_n = 0$

= u_n = $\limsup u_n = 0$

$\liminf u_n = 0$

= v_n = $\limsup v_n = 1$

$\liminf v_n = -1$

(d) Here we can see that

(s_n) does not converge as it alternates

(t_n) converges to 0.

(u_n) converges to 0.

(v_n) does not converge as the value oscillates between -1 and 1 for large values of n

(e) We need to see which are bounded.
(s_n), here the largest and smallest values are -1 and 1, so we have $[-1, 1]$.

It is bounded.

(ivn) Here this is bounded as it converges to 0 and has a largest value of 3/5. So we have [3/5, 0).

(vn), the largest and smallest values are 1/4 and −1/2, so we have [1/4, −1/2]. It is bounded

(vn) The largest and smallest values are 3/2 and −2/3, so we have [−2/3, 3/2], so it is bounded.

4. 11.5

(a). (r_n) is the enumeration of all rations [0, 1].

Here we can use the property that.

For any real number n, there exists a subsequence (r_{s_k}) that converges to n. Because there are infinitely many rational numbers in every interval, a subsequence r_{s_k} converges to n.

converges to a

So based on this property, we can say that $a \in [0,1]$, and from this we will get all the values.
Also we can have a sequence that converges to s, so we get $[0,1]$.

Answer: The set of all subsequential limits is $[0,1]$

(b) we need $\limsup a_n$ and $\liminf a_n$.

So here we have a set of subsequential limits.

So we know that
$\limsup a_n$ = sup (set of subsequential limits)

$= \sup [0,1] = 1$
$\limsup a_n = 1$
$\liminf a_n$ = inf (set of subsequential limits)

$= \inf [0,1]$

$= 0.$

$\liminf a_n = 0.$

Answer = $\limsup a_n = 1$

$\liminf a_n = 0$

5. Here we see that lim sup is the lim as $n \to \infty$ for all the values in the set of sup of (S_n). So it means the value the sup will reach after taking the sup for all S_n as $n \to \infty$.
Sup is only for a particular sequence, whereas limsup is the limit of all the sup values calculated from the sequence and subsequences.
A counter intuitive fact could be that:
It seems limsup would always be the maximum value in the set of possible values, but this is not always the case. This is not what lim sup is about.