Homework 4.

1. 12.10.

Here this is an if and only if so we need to prove both directions.

So first we can see that if (s_n) is bounded then $\lim\sup |s_n| < +\infty$.

So here as it is bounded
$$\sup\{|s_n|, n > N\} \leq b \text{ for } N.$$

So here $\lim_{n\to\infty} \sup\{|s_n|, n > N\} \leq b$

Here as it is bounded, we know $b < +\infty$

So here $\lim_{n\to\infty} \sup_n |s_n| \leq b < +\infty$

$$\text{So } \lim_{n\to\infty} \sup |s_n| < +\infty.$$

Now we need to check the other direction.

If $\limsup |s_n| < +\infty$, then (s_n) is bounded

so here $\lim_{n\to\infty} \sup |s_n| = L$

$$\lim_{n\to\infty} \sup\{ |s_n| , n > N \} = L$$

here we can say that $\varepsilon = 1$, and so.

$$|\sup\{|s_n|, n > N\} - L| < 1.$$

$$\sup\{|s_n|, n > N\} < 1 + L.$$

so here $|s_n| < 1 + L$ for all $n \geq N$.

so here we can say

$$n_0 = \max\{ |s_1|, \ldots, 1+L \}.$$

so here we have $|s_n| \leq n_0$ for all $n \in \mathbb{N}$.

so (s_n) is bounded.

so we have shown both directions of the implication.

Answer: We have shown that (s_n) is bounded if and only if $\lim\sup |s_n| < +\infty$

2. 12.12.

(a) Here we need to show that

$\lim\inf s_n \leq \lim\inf \sigma_n \leq \lim\sup \sigma_n \leq \lim\sup s_n$

we can show this inequality is parts.

we can first show that
$\lim\sup \sigma_n \leq \lim\sup s_n$.

so here we have $n > m > N$, and we see.

$$\sigma_n = \frac{1}{n}(s_1 + \cdots\cdots + s_n)$$

$$= \frac{1}{n}(s_1 + \cdots + s_N + s_{N+1} + \cdots s_n).$$

$$= \frac{1}{n}(s_1 + \cdots + s_N) + \frac{1}{n}(s_{N+1} + \cdots + s_n)$$

so here we can see that

$\frac{1}{n}(s_1 + \cdots + s_m) < \frac{1}{m}(s_1 + \cdots + s_m)$

and $\frac{1}{n}(s_{m+1} + \cdots + s_n) \leq \sup\{s_n, n > m\}$.

so here we see that.

$\sigma_n < \frac{1}{m}(s_1 + \cdots + s_m) + \sup\{s_n, n > M\}$.

So here we see that
$\sup\{\sigma_n, n > m\} \leq \frac{1}{m}(s_1 + \cdots + s_m) + \sup\{s_n, n > m\}$

$\lim_{m\to\infty} \sup\{\sigma_n, n > m\} \leq \lim_{m\to\infty} \frac{1}{m}(s_1 + \cdots + s_m) + \lim_{m\to\infty} \sup\{s_n, n > m\}$.

$\limsup \sigma_n \leq 0 + \limsup s_n$.

so $\limsup \sigma_n \leq \limsup s_n$.

so now we can look at a different part of the inequality.

now we can check that $\liminf s_n \leq \liminf \sigma_n$

Here as we know
$\limsup \sigma_n \leq \limsup s_n$.

we can use the property that
$\liminf \sigma_n \geq -\limsup(-\sigma_n)$.

so we can switch σ_n with $-\sigma_n$ and
s_n with $-s_n$.

So here we have that

$\liminf(-\sigma_n) \leq \liminf(-s_n)$

$-\limsup(-\sigma_n) \geq -\limsup(-s_n)$

so we get $-\limsup(-s_n) \leq -\limsup(-\sigma_n)$

Using the property, we can see that.

$\liminf s_n \leq \liminf \sigma_n$.

so we have proven this inequality.

we can now look at the

$\liminf \sigma_n \leq \limsup \sigma_n$.

So here we can say that

σ_{n_k} is a subsequence of σ_n. such that.

$\lim_{k\to\infty} \sigma_{n_k} = t.$

so here then we see that
$\liminf \sigma_{n_k} = \limsup \sigma_{n_k}.$

so here we can see that

$\liminf_{n\to\infty} \sigma_n \leq \liminf_{k\to\infty} \sigma_n.$

so $\liminf_{n\to\infty} \sigma_n \leq \limsup_{k\to\infty} \sigma_{n_k}$

$\limsup_{k\to\infty} \sigma_{n_k} \leq \limsup_{n\to\infty} \sigma_n$

so here we get $\liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n$

so with this we have proven all the inequalities

Answer> We have shown that
$\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n$

(b) so here we need to show that

lim σ_n exists and lim σ_n = lim s_n if lim s_n
exists

So here we can say lim $s_n = s$.

so we know that lim sup s_n = lim s_n = lim inf s_n
so lim sup $s_n = s$
lim inf $s_n = s$.

so we get the inequality as.

$$s \leq \liminf \sigma_n \leq \limsup \sigma_n \leq s$$

so here we have.
lim inf σ_n = lim sup σ_n = s.

so from this we get that
lim $\sigma_n = s$.

so here we see that lim $\sigma_n = s$ and
lim σ_n = s = lim s_n.
so lim σ_n = lim s_n.

Answer = We have proven that lim σ_n exists and
lim σ_n = lim s_n if lim s_n exists

(c) Here we need to give an example

so here we can say that

$s_n = (-1)^n + 1.$

so here we see that this is alternating and it does not converge.

now we can look at σ_n.

Here we can see that

$$\sigma_n = \begin{cases} \frac{1}{n}(n) = 1, & \text{when } n \text{ is even} \\ \frac{1}{n}(n-1) = 1 - \frac{1}{n} & \text{when } n \text{ is odd} \end{cases}$$

so $\lim \sigma_n = \lim_{n\to\infty} 1 = 1$

$\lim_{n\to\infty} 1 - \frac{1}{n} = 1 - 0 = 1.$

so we get $\lim \sigma_n = 1 = 1$

so $\lim \sigma_n = 1.$

so here we have $\lim a_n = 1$ and s_n does not have a limit.

Answer = Here we have $\lim a_n$ exists, but $\lim s_n$ does not exist

3. 14.2

(a) $\sum \frac{n-1}{n^2}$.

Here $a_n = \frac{n-1}{n^2}$

we can see that

$$\frac{n-1}{n^2} > \frac{1}{2n}.$$

so here $\sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n}$.

$\sum \frac{1}{n}$ diverges.

so here we see that $\sum \frac{n-1}{n^2}$ diverges,

Answer = It diverges

(b) $\sum (-1)^n$.

Here $b_n = 1$, as $a_n = (-1)^n \cdot b_n$.

so here we need b_n to be decreasing.

Here we see b_n is decreasing as
$b_{n+1} \leq b_n$ as $1 \leq 1$.

We need $\lim_{n\to\infty} b_n = 0$

$\lim_{n\to\infty} 1 \neq 0$, $1 \neq 0$.

So we see that it does not converge.
It diverges.

Answer = It diverges.

(c) $\sum \frac{3n}{n^3}$.

$$= 3 \sum \frac{n}{n^3}$$

$$\sum \frac{n}{n^3} = \sum \frac{1}{n^2}.$$

Here we have a $3 \cdot \sum \frac{1}{n^2}$

we know that $\sum \frac{1}{n^2}$ converges, so

$3 \cdot \sum \frac{1}{n^2}$ also converges.

it converges.

Answer $\Rightarrow$ it converges.

(d). $\sum \frac{n^3}{3^n}$.

Here we can use the ratio test.

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)^3}{n} \cdot \frac{3^n}{3^{n+1}} \right|.$$

$$= \frac{1}{3} \lim \left| \left(1 + \frac{1}{n}\right)^3 \right|$$

$$= 1 \lim(1) = 1.$$

$\frac{\cdot}{3}$ $\qquad$ $\frac{\cdot}{3}$

Here $\frac{1}{3} < 1.$

So it converges.

we have that it converges.

Answer = it converges

(e) $\sum \frac{n^2}{n!}$

we can use the ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2} \right|$$

$$= \lim \left| \frac{(n+1)^2}{(n+1)} \cdot \frac{1}{n^2} \right|$$

$$= \lim \left| \frac{n+1}{n^2} \right|$$

$$= \lim \left| \frac{n(1+1/n)}{n^2} \right|$$

$$\lim \left(\frac{1}{n} + \frac{1}{n^2} \right)$$

$$= \lim \frac{1}{} + \lim \left(\frac{1}{}\right)^2$$

$\lim_{n\to\infty} \frac{}{n} \qquad \lim_{n\to\infty}\left(\frac{}{n}\right)$

$= 0 + 0$

$= 0.$

So as $0 < 1$, it converges

It converges

Answer = It converges

(f). $\sum \frac{1}{n^n}$.

we can use the root test

$|a_n|^{1/n} = \left(\frac{1}{n^n}\right)^{1/n}$

$= \frac{1^{1/n}}{n^{(1/n)}}$

$\lim_{n\to\infty}\left|\frac{1^{1/n}}{n}\right|$

$= \lim_{n\to\infty}\left|\frac{1}{n}\right|$

$= 0$

Here it converges as $0 < 1$.

So it converges.

Answer = It converges.

(g). $\sum \frac{n}{2^n}$.

Here we can do the ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)}{2^{n+1}} \cdot \frac{2}{n}\right|$$

$$= \left|\frac{n+1}{2n}\right|$$

$$\lim \left|\frac{n+1}{2n}\right| = \frac{1}{2} \lim \left|\frac{n+1}{n}\right|.$$

$$= \frac{1}{2} \cdot \lim_{n\to\infty} 1 + \frac{1}{n}$$

$$= \frac{1}{2} \cdot 1 = \frac{1}{2}$$

$\frac{1}{2} < 1$, so it converges.

It converges.

Answer = It converges

4. (4,1)

We need a series which diverges by the root test but gives no information for the ratio test.

$$a_n = ((-1)^n+2)^n.$$

Here we can check the root test.

$$|a_n|^{1/n} = (-1)^n+2.$$

$$\limsup((-1)^n+2)$$

if n is even, we get

$$1+2=3$$

if n is odd, we get

$$-1+2=1$$

So here we get subsequential limits as

$\{1, 3\}$.

So $\sup S = 3$.

Here $3 > 1$, so it diverges by the root test.

Now we can check the ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{((-1)^{n+1}+2)^{n+1}}{((-1)^n+2)^n}\right|.$$

If n is even, we get.
$n = 2k$.

$$= \left|\frac{((-1)^{2k+1}+2)^{2k+1}}{((-1)^{2k}+2)^{2k}}\right|.$$

$$= \left|\left(\frac{1}{3}\right)^{2k}\right|$$

So here $\lim_{k\to\infty}\left|\left(\frac{1}{3}\right)^{2k}\right| = 0.$

If n is odd, we get.

$$\left|\frac{((-1)^{2k+2}+2)^{2k+2}}{\ldots}\right|$$

$$\left|\frac{((-1)^{\dots} + 2)^{\dots}}{((-1)^{2k+1}+2)^{2k+1}}\right|$$

$$= \left|3^{2k+2}\right|$$

$$\lim_{k\to\infty} \left|3^{2k+2}\right|$$

$$= \infty.$$

so we get the set of subsequential limits as.

$$S = \{0, \infty\}.$$

so here $\inf(S) = 0 < 1$.

and $\sup(S) = \infty > 1$.

so here $\liminf(a_{n+1}/a_n) < 1 < \limsup(a_{n+1}/a_n)$

so we get no information.

So this is a series $\sum a_n$ that satisfies the condition.

Answer : We have shown a series $\sum a_n$ that satisfies the conditions

Rudin.

5. 6.

(a) $a_n = \sqrt{n+1} - \sqrt{n}$

Here we can do.

$$\frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} - \sqrt{n})}{\sqrt{n+1} - \sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

Now we can see that.

$$\sqrt{n+1} + \sqrt{n} \leq \sqrt{n+1} + \sqrt{n+1}$$

so as we know that

$\frac{1}{2} \cdot \frac{1}{\sqrt{n+1}}$ diverges, we can see

by the comparison theorem that

$\frac{1}{\sqrt{n+1}+\sqrt{n}}$, $\frac{1}{2} \cdot \frac{1}{\sqrt{n+1}}$, so we

see that $\frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges.

Answer = ☆ diverges

(b). $\frac{\sqrt{n+1} - \sqrt{n}}{n}$

we can multiply this to get

$$\frac{\sqrt{n+1}-\sqrt{n}}{n} \cdot \frac{(\sqrt{n+1} - \sqrt{n})/n}{(\sqrt{n+1} - \sqrt{n})/n}$$

$$= \frac{1}{n(\sqrt{n+1}+\sqrt{n})}$$

Now here we can see that

$$n(\sqrt{n+1}+\sqrt{n}) \to n^{3/2}.$$

so here $\frac{1}{n(\sqrt{n+1}+\sqrt{n})} \leq \frac{1}{n^{3/2}}$.

Here as $3/2 > 1$, we know that $\sum \frac{1}{n^{3/2}}$ would converge.

So here we have.

$\frac{\sqrt{n+1} - \sqrt{n}}{n}$ would converge.

Answer = It will Converge.

b. 7.

Here we need to prove that.

$\sum \frac{\sqrt{a_n}}{n}$ Converges if $\sum a_n$ converges.

So here we can assume that $a_n \geq 0$ $\forall n$.

So here we can see that as $\sum a_n$ converges.

we can say that $\frac{\sqrt{a_n}}{n} \geq b_n$

So b_n would converge if

$b_n < a_n$.

Here we can see that
$\frac{\sqrt{a_n}}{n} < a_n$ as here $n > 0$ and

$a_n \geq 0$.

So here by the comparison test we see that $\sum \frac{\sqrt{a_n}}{n}$ will converge.

It will converge.

Answer = It will converge.

7. 9.

(a) $\sum n^3 z^n$.

Here we need to find the radius of convergence.

$$\lim \frac{a_n}{a_{n+1}} = \frac{n^3}{(n+1)^3}$$

$$= \lim n^3 = 1.$$

$$n\to\infty \quad \overline{(n+1)^3}$$

So the radius of convergence is 1.

Answer = 1.

(b) $\sum \frac{2^n}{n!} z^n$.

So $a_n = \frac{2^n}{n!}$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{2^n}{n!} \cdot \frac{(n+1)!}{2^{n+1}}$$

$$= \frac{n+1}{2}$$

$$\frac{1}{2} \cdot \lim_{n\to\infty} n+1 = \frac{1}{2} \cdot \infty = \infty.$$

So the radius of convergence is ∞ !!

Answer = ∞ !