Homework 5

1. 13-3

(a) we need to show that d is a metric for B.

so here we need to show that
$d(x_i, x_i) = 0$
$d(x_i, y_i) = d(y_i, x_i)$.
$d(x, z) \leq d(x, y) + d(y, z)$.

so here we see that
$d(x, y) = \sup\{|x_j - y_j|, j = 1, 2, \ldots\}$.

Here $d(x, x)$, we get
$d(x, x) = \sup\{|x_j - x_j|, j = 1, 2, \ldots\}$

$= \sup\{0, 0, \ldots\} = 0$.

so $d(x, x) = 0$
Now we need to see if $d(x_i, y_i) = d(y_i, x_i)$.

$d(x_i, y_i) = \sup\{|x_j - y_j|, j = 1, 2, \ldots\}$.

so we can say that we get the largest difference
at position i , for $i \in \{1, 2, \ldots\}$
so we have or assume.
$\sup\{|x_j - y_j|\} = |x_i - y_i|$.

Now we can look at $d(y, x)$
$= \sup\{|y_j - x_j|, j = 1, 2, \ldots\}$.

so here we know the largest absolute difference and that $|x_i - y_i|$ is the largest also $|y_i - x_i| = |-(x_i - y_i)|$
$= |(x_i - y_i)|$
So here we can see that position i is still the sup as $|x_i - y_i| = |-(y_i - x_i)| = |y_i - x_i|$.
so $d(x, y) = d(y, x)$.

Now we need to check $d(x, z) = d(x, y) + d(y, z)$

so here we have
$d(x, z) = \sup\{|x_j - z_j|, j = 1, 2, \ldots\}$.

we can say that

$\sup\{|x_j - z_j|\} = \sup\{|x_j + y_j - y_j - z_j|\}$

so we can use the triangle inequality to see that

$\sup\{|x_j + y_j - y_j - z_j|\} \leq \sup\{|x_j - y_j|\} + \sup\{|y_j - z_j|$

for $j = 1, 2, \ldots$

so $d(x,z) \leq d(x,y) + d(y,z)$,

so we get that $d(x,z) \leq d(x,y) + d(y,z)$

so we have shown all the conditions.

Answer = we have shown that d is a metric for B.

(b). we need to see if d^* is a metric for B.

so here we know that d must be a non infinite value.

However, here we see that if

$x = (1, 1, 1, \ldots \ldots 1)$

$y = (0, 0, 0, \ldots \ldots, 0)$

we get $d(x,y) = \sum_{j=1}^{\infty} |1-0| = \sum_{}^{\infty} 1 = \infty$.

Also if $y = (1, \ldots, 1)$
$x = (0, \ldots, 0)$
we get $\sum_{j=1}^{\infty} |0-1| = \sum_{j=1}^{\infty} 1 = \infty$.

So as the metric is infinite, we can see that $d^*(x,y)$ is not a metric for B.

so it is not a metric for B

Answer: $d^*(x,y)$ is not a metric for B.

2. 13.5

(a) we need to verify one of De Morgan's laws
we can show this by showing that each are subsets of each other
so here we have.
$\cap\{S \setminus U, U \in \mathcal{U}\} = S \setminus \cup\{U, U \in \mathcal{U}\}$.

so here we can say that

$x \in \cap\{S \setminus U, U \in \mathcal{U}\}$

so here we know that
$x \in S \setminus U$ for all $U \in \mathcal{U}$.

so as $S \setminus U$. means values outside of U, we get that $x \notin U$ for all $U \in \mathcal{U}$.

so $x \in \bigcup \{U, U \in \mathcal{U}\}$.

so here we get that.

$x \in S \setminus U = x \in S \setminus \bigcup \{U, U \in \mathcal{U}\}$.

so we have shown that

$\bigcap \{S - U, U \in \mathcal{U}\} \subseteq S \setminus \bigcup \{U, U \in \mathcal{U}\}$.

Now we can check the other direction

so if $x \in S \setminus \bigcup \{U, U \in \mathcal{U}\}$

Then we know that
$x \notin \bigcup \{U, U \in \mathcal{U}\}$.
As this is the case for all U, we can see that.

$x \in S \setminus U$

So here $x \in \cap\{S\setminus U, U \in u\}$.

so we have $S\setminus \cup\{U, U\in u\} \subseteq \cap\{S\setminus U, U\in u\}$.

so here we have that.

$$S\setminus \cup\{U, U\in u\} = \cap\{S\setminus U, U\in u\}.$$

Answer = we have verified one of DeMorgan laws for sets

(b) we need to show that the intersection of any collection of closed sets is a closed set.

so here we can say that.

so here we have.

$$\cap\{U, U\in u\} = S\setminus \cup\{S\setminus U, U\in u\}.$$

so as U is a closed set, $S\setminus U$ is a open set.

so here $S\setminus \cup\{S\setminus U, U\in u\}$ is a collection of open sets.

So if $\{U_i, U \in u\}$ is a collection of open sets then we can say.

for $s_0 \in \bigcup_{i \in U} U$.

So there exists some $i \in U$, such that $s_0 \in U_i$.

Since U_i is an open set, we know that there exists a number $r > 0$ such that

$\{t \in S, d(t,s) < r\} \subseteq U_i$.

So as $U_i \subseteq \bigcup U_i$ we see that.

$\{t \in S, d(t,s) < r\} \subseteq \bigcup U_i$

So here we see that $\bigcup U_i$ is an open set. So that would mean that.

$S \setminus \bigcup \{S \setminus U, U \in u\}$ is a closed set

So the set $\bigcap \{U, U \in u\}$ is a closed set.

Answer: we have shown this proof.

3. 13.7.

Here we need to show that every open set in $\mathbb{R}$ is the disjoint union of a finite or infinite sequence of open intervals.

So here we can say that O is an open set in $\mathbb{R}$.

So for each $s \in O$, we know there is a positive number such that $\varepsilon > 0$ and

$$\{t, d(t,s) < \varepsilon\} \subseteq O.$$

So we see that

$$O = \bigcup_{s \in O} \{t, d(t,s) < \varepsilon\}.$$

So here we can substitute this equation to get

$$(s-\varepsilon, s+\varepsilon)$$

So we see that

$$O = \bigcup_{s \in O} (s-\varepsilon, s+\varepsilon)$$

So we can say that X_s is the longest interval in O that contains s..

So then from this we get that.

$$O = \bigcup_{s \in t} X_s.$$

So here we can assume that

$X_s \cap X_t \neq \phi$, so then we see that $X_s \cup X_t$ must be an open interval.

So as $X_s \subseteq X_s \cup X_t$ and $X_t \subseteq X_s \cup X_t$, we get that $X_s = X_s \cup X_t$.

So this would mean that

$X_t \subseteq X_s$, and we can see the other direction where $X_s \subseteq X_t$.

So we get that $X_s = X_t$.

So we get that $X_s = X_t$ or $X_s \cap X_t = \phi$.

So this would give us that O is the disjoint union of finite or infinite open intervals.

Answer = we have proven the statement.

4. So here we need to show that if

$S_1 = \bar{S}$ and $S_2 = \bar{S_1}$, then $S_1 = S_2$.

So firstly we know that $S \subseteq \bar{S}$ and so we can say that $\bar{S} \subseteq \bar{\bar{S}}$

So here $S_1 = \bar{S}$, $S_2 = \bar{S_1}$.
so $S_2 = \bar{\bar{S}}$ so if we show that $\bar{S} = \bar{\bar{S}}$ then we can prove this statement.

Here we already know that

$\bar{S} \subseteq \bar{\bar{S}}$ so now we would need to take the other direction.
we need to show that $\bar{\bar{S}} \subseteq \bar{S}$.

So here we know that for each $a \in \bar{\bar{S}}$, there is some sequence that we have in $\bar{S}$ that $\lim \bar{s}_n = a$. so we also know that for each $b \in \bar{S}$, we have another sequence in S that is $\lim s_n > b$.

so we have some N, such that if $k > N$, we have $d(\bar{s}_n, s_n) < \frac{1}{n}$ for all $n > 0$.

so we can find a sequence s in S for the $s = s_n$ for some $k > k$.

So we now see that the $\lim s = a$. we can show this as we see

$$d(s, a) \leq d(\bar{s}_n, s) + d(\bar{s}_n, a)$$
$$= d(s, \bar{s}_n) + d(\bar{s}_n, a) < \varepsilon.$$

so here we get $d(s, a) < \varepsilon$, so this means that $\lim s = a$.
So as s is a sequence in S, we see that $a \in \bar{S}$, so this would mean that

$\bar{\bar{S}} \subseteq \bar{S} = \bar{S} \subseteq S$. so

from this as we have
$S \subseteq \bar{S}$ and now $\bar{S} \subseteq S$, we
get $= \bar{S} = S$ and so
$\bar{S} = \bar{\bar{S}}$, so this statement is
true

Answer = we have shown that this statement is true

5. Now we need to prove that $\bar{S}$ is the intersection of all closed subsets in X that contains S.

So we can call I the intersection of all closed subsets that contains S.

So firstly as we see that I is the intersection of closed sets, we know that I must also be closed. Now we can also see that as $\bar{S}$ is a closed set and that it contains S, we can see that

$I \subseteq \bar{S}$.

From part 4, we know that since I is closed $\bar{I} = I$.

So as $S \subseteq I$, we see that all sequential limits will be in I.

So then $S \subseteq \bar{I}$.

So here $\bar{S} \subseteq I$, so from

this we can see that

$\overline{S} = \overline{I}$, so this would mean that $\overline{S}$ is the intersection of all closed subsets in X that contains S.

Answer = We have proven this statement. //