Homework 6.

1. we need to show that if $[0,1]$ is sequentially compact, then $[0,1]$ is $\mathbb{R}^2$ i.e. sequentially compact.

so here we can say that if X and Y are sequentially compact, then $X \times Y$ is also sequentially compact.

So we have $(x_n, y_n) \in X \times Y$. we know.

that $x_n \in X$ and that $(x_n, n \in \mathbb{N})$ has a convergent subsequence $\{x_{n_k} \mid k \in \mathbb{N}\}$ is X with limit a.

so we have $\{y_n \mid n \in \mathbb{N}\}$ and we have a convergent subsequence $\{y_{n_{k_j}} \mid k \in \mathbb{N}\}$ with a limit b.

So now we see that $(x_{n_{k_j}}, y_{n_{k_j}})$ converges to (a, b) in $X \times Y$.

So now we get that $X \times Y$ is sequentially

compact, we can see that
if $X = R$ and $Y = R$, and a subsequence of X and Y converges is $[0,1]$.
Then $X \times Y = R \times R = R^2$.
So then R^2 converges such that if $x_{n_{k_j}}$ converges to a, where $a \in [0,1)$ and $b \in [0,1)$ for $y_{n_{k_j}}$ we see that $(a,b) \in R^2$. So from this we see that $[0,1]$ in R^2 is sequentially compact.

Answer= we ~~have proven the statement~~

2. Here we need to see if E is countable and if E is compact.

So here we can see or say that

$$E = \bigcap_{n=1}^{\infty} E_n \quad \text{where}$$

$$E_n = \bigcup_{s \in S_n} \left[\frac{a}{10^n}, \frac{a+1}{10^n} \right]$$

we use this range as we know $x \in [0,1]$.

So here we see all the decimals for the

first n places. so the $\cap E_n$ gives us all the possible values for each n, so for this reason we can use this formula.

Now we can see that E_n is closed as it is the union of closed intervals. so as E_n is closed we can say that E must be closed.

so as E is closed and as E is the set of $x \in [0,1]$, we know the elements is E are bounded for $[0,1]$, so E is also bounded.

so from this bounded and closed property of E, we can say that E is compact.

Now we can check if E is countable.

in order for it to be countable, we would need an injective function from $[0,1] \to E$.

so the here mapping can be done over the fraction.

So we can have a mapping such that the map is injective when any two sequences differ in at least one place and we get two different numbers.

So from this we can get that E is uncountable as we would not be able to get a finite number of values

So as E is not empty as $\exists a \in E$, we can say that E is uncountable.

Answer = we have shown the statement.

3. Here we need to show that if $B = \bigcup_i A_i$, then $\bar{B} \supset \bigcup_i \bar{A_i}$.

So here as $B = \bigcup_i A_i$, we know that

$A_i \subset B$ for all i.

So as $A_i \subset B$ for all i, we can say that $\bar{A_i} \subset \bar{B}$ for all i.

so from this we can get that

$\cup_i \overline{A_i} \subset \overline{B}$.

now we can check if it is a strict ~~inclusion~~

so here we can take an example of an infinite union like.

$(0,1) = \cup_i (\frac{1}{n}, 1)$.

Here this is true as $(\frac{1}{n}, 1) \in (0,1)$ as $n \to \infty$, $\frac{1}{n} \to 0$, so we see that this is ~~true~~

$\overline{(0,1)} = [0,1] \supset \cup_i \overline{(\frac{1}{n}, 1)}$

$\cup_i \overline{(\frac{1}{n}, 1)} = \cup_i [\frac{1}{n}, 1]$

$\cup_i [\frac{1}{n}, 1] = (0,1]$.

so we see that we get

$\overline{(0,1)} = [0,1] \supset (0,1]$.

So we can see that the inclusion is not a strict inclusion.

Answer: we ~~have~~ ~~proven the statement~~.

4. Here we need to give a example of a closed set which is not a countable union of closed intervals.

For the counterexample, we could use the same set E from the second question.

So we can say that we have no closed interval where $[a,b] \subseteq E$.

But we get a contradiction for this as for some n we see that we get $a + \frac{1}{n} \in E$.

So as the nth digit of a is either 4, 7 or 0, we see that $a + \frac{1}{n} \in \{1, 5, 8\}$.

However, we know that E has only 0, 4, 7 so we see that {1, 5, 8} are not in E.

So if we write E as an union of closed intervals then it would be the union of intervals where each interval has only 1 value.

Here as E is uncountable, we get an uncountable union of closed intervals.

We can see that the argument is wrong where it says that it is a union of closed intervals.

We see that this argument will not show that E is the union of closed intervals as it is a result of intersections.

So this would be the problem with the argument.

Answer = we have proven ~~and shown the statement~~