Homework 7.

1. If X and Y are open cover compact, then $X \times Y$ is open cover compact.

Here we know that X and Y are open cover compact. so it means that each of its open covers has a finite subcovers

$$X = \bigcup_{x \in C} x.$$

we know there is a finite subcollection given by $X = \bigcup_{x \in F} x.$

Now we need to show that $X \times Y$ is compact.

so here as X and Y are compact, we can have an open cover for $X \times Y$ as Z.

Now we can say that $A \subseteq X$ is good if $A \times Y$ is covered by a finite subsets of Z. Now we need to show that X is good.

so we can assume that $A_1, \ldots\ldots, A_n$ are good.

so $A = \bigcup_{i=1}^{n} A_i$

so for any i, we have $A_i \times Y$ is covered by a finite subset of Z, so we can see that $Z_i \subseteq Z$.

so here $A \times Y = \bigcup_{i=1}^{n} A_i \times Y$.

so this is covered by the finite subset $\bigcup_{i=1}^{n} W_i$.

Now we can first see that X is locally satisfied.

So we need to show that $\forall x \in X$, there is an open set $U(x)$ such that $x \in U(x)$ and $U(x)$ is good.

so now we can chose $x \in X$.

For $y \in Y$, we have $(x, y) \in Z$, as Z is $(X \times Y)$ covered.

So we see that

$(x, y) \in U(x) \times V(y) \subseteq Z(y)$.

$V(y)$ is an open cover for Y

so as T_2 is compact, we know that $V(y)$ has a finite subcover.

we can say that

$$U(x) = U(y_1) \cap \cdots\cdots \cap U(y_n)$$

so here we have. $U(x) \times V(y_i) \subseteq U(y_i) \times V(y_i)$

so we can see that.

$$U(x) \times Y = U(x) \times \bigcup_{i=1}^{n} V(y_i)$$

$$\subseteq W(y_i)$$

so as $U(x) \times Y \subseteq W(y_i)$, we can see that for $x \in U(x)$ and $U(x)$ is open in X.

Now we can see that for each $x \in X$, we have. $U(x)$ as a valid open set.

so as $U(x)$ is an open set, then we know that the open cover is a collection of open sets so here. we see that

the open cover is $\{U(x), x \in X\}$.

So as we know that X is compact and $U(x)$ has a finite subcover, we get that X is also good and valid.

As X is valid, we can see that we get $X \times Y$ as an open cover compact

Answer = we have proven ~~the statement~~

2. (a): If A is open, then $f(A)$ is open.

we can see that if we have $f(x) = c$, where c is a constant, then for any x, we get c.

So then $\forall a \in A$ $f(a) = c$.
so the output set is $\{c\}$. so this would mean that this is closed.

So $f(A)$ is closed. So the statement is false.

Answer = This statement is ~~False~~

(b): If A is closed then $f(A)$ is closed

Here we can take $f(x) = x^4$.
Here we can see that.
~~X~~ is $\mathbb{R}$ and so we know that it is closed.
So as X is closed, we know that A is closed.

Here we see that
for $f(X)$, we get, $[0, \infty)$.

So this is open as no open ball with 0 is contained.
So it is False.

Answer = This statement is False.

(c). A is bounded, then $f(A)$ is bounded.
Here we can take f as -
$f : (0,1) \to \mathbb{R}$.
where $f(x) = \frac{1}{x}$.

Here we see that.
for, $f(x) \in (1, \infty)$.

So this is not bounded.

so. this is False

Answer = This statement is False

(d). If A is compact, then $f(A)$ is compact.

so here we can take A as the open cover of $f(A)$, then we can say that.

$B = \{ f^{-1}(a) \mid a \in A \}$.

so here B is an open cover of A.

so we can get a finite subcover of A from B to obtain the finite subcover $\{A_1, \ldots, A_n\}$ from A to $f(a)$.

so this would give us that $f(A)$ is compact.

so this is true.

Answer = The statement is true

(e). If A is connected, then $f(A)$ is connected.

so here. we can see that we can show if X is connected, then Y is connected.

so let X be connected. Here we can say that Z is a subset of Y which is both open and closed. so we get that

$f^{-1}(Z)$ is also open and closed in X.

so connectedness of X gives us that either $f^{-1}(Z) = \phi$ or $f^{-1}(Z) = X$.

so as f is onto, we can see that $f^{-1}(Z) = \phi$, so we get $Z = \phi$ and $f^{-1}(Z) = X$ giving us $Z = Y$.

so we get that Y is connected, so $f(A)$ is connected

It is true...

Answer = This is true

3. Here we need to show that we cannot get a Continuous map $f: [0,1] \to \mathbb{R}$ such that f is surjective.

So here we have $f: [0,1] \to \mathbb{R}$.

We can see that $[0,1]$ is compact.

So we know that as A is compact, $f(A)$ is compact.

So here $f(x)$, $x \in [0,1]$ is compact.

So here this would indicate that $\mathbb{R}$ would have to be compact.

However, we know that $\mathbb{R}$ is not compact. So we get a contradiction.

So we see that we cannot get a continuous map f, such that f is surjective.

So we have shown this statement.

Answer: we have proven this statement.