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Homework 8.

1. Here we need to show that f_n converges uniformly on $\mathbb{R}$.

$$f_n(x) = \frac{n + \sin x}{2n + \cos n^2 x}$$

$$f(x) = \sum_{n=0}^{\infty} f_n(x).$$

Here we can see that as $\lim_{n\to\infty} \frac{n + \sin(x)}{2n + \cos n^2 x} = \frac{n}{2n} = \frac{1}{2}.$

So we can check that if $f(x) = \frac{1}{2}$

then f_n converges uniformly.

So here we have.

$|f_n(x) - f(x)| \leq \varepsilon.$

So here we can check that

$$\left|f_n(x) - \frac{1}{2}\right| \leq \varepsilon.$$

$$= \left|\frac{n+\sin x}{2n+\cos n^2 x} - \frac{1}{2}\right| = \left|\frac{2n+2\sin x - 2n - \cos n^2 x}{4n + 2\cos n^2 x}\right|$$

$$= \left|\frac{2\sin x - \cos n^2 x}{4n+2\cos n^2 x}\right|$$

so here we can see that.

$$\left|\frac{2\sin x - \cos n^2 x}{4n+2\cos(n^2 x)}\right| \leq \frac{3.}{4n-2.}$$

so here we get that.

for $\varepsilon > 0$ and $\varepsilon > \frac{3}{4n-2}$, we have $n > N$, such that.

$$\left|f_n - \frac{1}{2}\right| < \varepsilon.$$

Here we can say that f_n converges uniformly to $\frac{1}{2}$.

So f_n converges uniformly on $\mathbb{R}$.

Answer = we have shown that f_n converges uniformly on $\mathbb{R}$

2. we need to show that the series is continuous on $[-1, 1]$ if $\sum |a_n| < \infty$.

$$f(x) = \sum_{n \geq 1}^{\infty} a_n x^n.$$

So here $f_n(x) = a_n x^n$.

So we can see that $f_n(x)$ to $f(x)$ converges uniformly

So as $x \in [-1, 1]$, we can see that

$|f_n(x)| = |a_n x^n| = |a_n| \, |x|^n$

Here as $x \in [-1, 1]$, we can see that

$|f_n(x)| \leq |a_n|$

So here we see that the series $\sum f_n(x)$ will converge uniformly

Now we know that as each f_n converges we see that as $\sum f_n(x)$ converges for each point, to f, we see that the summation will also be uniformly convergent.

now we know that each $f_n(x)$ is continuous and we know that the sum of continuous function is continuous.
$\sum f_n(x)$ is continuous so.
f would be continuous as the partial sums are continuous and converge to f.

now we need to show that $\sum_{n \geq 1}^{\infty} n^{-2} x^n$ is continuous on $[-1,1]$.

so here we can see that.

$$f_n(x) \leq \left| \frac{1}{n^2} \right| \leq \frac{1}{n^2}.$$

so now we know that $\sum f_n(x) \leq \sum \frac{1}{n^2}$.

we know that $\sum \frac{1}{n^2}$ converges so.

we can see that $\sum n^{-2} x^n$ would also converge. Also as this is continuous for each term the partial sums would be continuous and so we

get that f converges and continuous.
when $x \in [-1,1]$.

Answer = we ~~have shown the statements~~

3. Here we need to show that.

$f(x) = \sum_n x^n$ is continuous on $(-1,1)$

so here $f_n = x^n$.

so $|f_n| = |x^n|$
for $x \in [-a,a]$, $0 \le a < 1$, we have.

$|f_n| = |x|^n \le a^n$.

so here we see that $\sum a^n$
converges when $0 \le a < 1$

So as each (f_n) converges, we have.
that $f = \sum f_n(x)$ is also convergent.

so we have shown that $f(x)$ on $(-1,1)$
is continuous by showing uniform convergence

for $0 < a < 1$, for $x \in [-a, a]$
as $\sum a^n = \frac{1}{1-a}$

so $f(x)$ is continuous. Now we can show that it is not uniform convergence.

Now here we can see that $f_n(x)$ is bonded for $x \in (-1, 1)$ as x^n is bonded for these values.

So we know that the partial sum of bonded values is bonded.

But, we can see that $f(x)$ is unbonded as for any $n > 0$, we have

$$f\left(1-\frac{1}{n}\right) = \frac{1}{1-\left(1-\frac{1}{n}\right)} = n.$$

So we can get any n for $n > 0$.
so this would indicate that $f(x)$ is unbonded.

so we get a contradiction.
So this would mean that as f_n is

bounded and $f(x)$ is unbounded, then

$f_n(x)$ does not converge uniformly to $f(x)$

So we have shown that it is continuous

on $(-1,1)$ but it is not uniformly convergent.

Answer = We have shown both the statements