Homework 9.

1. Here we need a smooth function $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x \geq 1$ and $f(x) \in [0,1]$ when $x \in [0,1]$.

So we can use the example in Ross to get.

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

So now we can use $f(x)$ and $f(1-x)$ to get the equation given by.

$$g(x) = \frac{f(x)}{f(x) + f(1-x)}$$

So now we need $g(x)$ to be smooth. Also we can see that when $x \geq 1$, $g(x) = 1$, when $x \leq 0$, $g(x) = 0$. when $x \in [0,1]$, $g(x) \in [0,1]$.

Now for $g(x)$ to be smooth both the numerator and denominator need to be smooth or infinitely differentiable.
we know $f(x)$ is infinitely differentsable.

we can check $f(1-x)$.
$$\frac{d^n f(1-x)}{dx^n} = (-1)^n \cdot f^{(n)}(1-x),$$
So this is infinitely differentiable.

Also $f(x) + f(1-x)$ is infinitely differentiable as it is the sum of infinitely differentiable function.

So this would mean that $g(x)$ is infinitely differentiable, so it is smooth.

So we get $g(x) = \frac{f(x)}{f(x)+f(1-x)}$ that satisfies the conditions.

Answer = the smooth function is $g(x) = \frac{f(x)}{f(x)+f(1-x)}$

2. 5.4

Here if $c_0 + \frac{c_1}{2} + \cdots + \frac{c_{n-1}}{n} + \frac{c_n}{n+1} = 0.$

we need to see that $c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} + c_n x^n = 0$ has at one real root between 0 and 1.

So here to solve this, we can create another polynomial given by.

$$f(x) = c_0 x + \frac{c_1}{2}x^2 + \cdots\cdots + \frac{c_{n-1}}{n}x^n + \frac{c_n}{n+1}\cdot x^{n+1}.$$

now after getting this, we can take the derivative of this function.

$$f'(x) = \frac{d}{dx}\left(c_0 x + \cdots + \frac{c_n}{n+1}\cdot x^{n+1} \right),$$

$$= c_0 + c_1 x + \cdots + c_n x^n.$$

Now we can see that $f'(x)$ is equal to the function or polynomial given in the question.

So now we can look at Rolles theorem.

Here we know that if f is continuous on $[a,b]$ and differentiable on (a,b), and $f(a) = f(b)$, then there exists at least one c in the open interval (a,b) such that

$f'(c) = 0$.

So now when $f'(c) = 0$, we know that c is a root of the polynomial.

So here we can take $[a,b]$ as $[0,1]$.

So here $f(0) = c_0 \cdot 0 + \cdots \cdots + \frac{c_n}{n+1} \cdot 0^{n+1}$
$= 0$.

$f(1) = c_0 \cdot x + \frac{c_1}{2} \cdot x^2 + \cdots + \frac{c_n}{n+1} \cdot x^{n+1}$

$= c_0 \cdot 1 + \cdots \cdots + \frac{c_n}{n+1} \cdot 1^{n+1}$

$= c_0 + \cdots \cdots + \frac{c_n}{n+1} = 0$.

So $f(0) = f(1) = 0$.

So as we know this is true, we know that

$f'(c) = 0$ for some $c \in (a,b)$, where

$c \in (0,1)$.

So this would mean that

$f'(c) = c_0 + \cdots + c_n \cdot c^n = 0.$

So here we can say that now $x = c$ and so.

As we have at least one where $c_0 + \cdots + c_n \cdot c^n = 0$, we can see that we have at least one real root between 0 and 1.

Answer = we have proven the ~~statement~~

3. 5.8.

we need to show that $\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$

So here we need to convert

$\frac{f(t) - f(x)}{t - x}$ to a differential.

so here we know that this is true whenever.

$0 < |t - x| < \delta$.

So here we have $x, t \in [a, b]$.

So we can say that we can let δ be such that. $|f'(y) - f'(x)| < \varepsilon$, whenever.

$y, x \in [a, b]$. and $|y - x| < \delta$.

So then we know that there exists a y such that we can see.

$$\frac{f(t) - f(x)}{t - x} = f'(y).$$

So now we have used the mean value theorem to get a differential.

So we now have.

$|f'(y) - f'(x)| < \varepsilon$.

so as we know that for all $y, x \in [a, b]$, we know that.

as. $|y - x| < \delta$, we can see that.
we can satisfy the condition that
$|f'(y) - f'(x)| < \varepsilon$.

So now we can combine these to get.

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| = |f'(y) - f'(x)|$$

so we know that $|f'(y) - f'(x)| < \varepsilon$.

So this would mean that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon.$$

so we have shown the statement.

Answer = we have proven the statement

5.18.

we need to derive that

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!} \cdot (\beta - \alpha)^n$$

so from the question we know that $Q(t)$ is not differentiable at $t = \beta$.

so we can see that we would need to differentiate Q $(n-1)$ times based on the equation.

so we know that

$$f(t) - f(\beta) = (t - \beta) Q(t)$$

so $f(t) = (t - \beta) Q(t) + f(\beta)$.

so by induction, we know that $f(t) = (t - \beta) Q(t) + f(b)$.

By induction we can see that we have the equation given by.

$$f^{(k)}(t) = (t - \beta)\, Q^{(k)}(t) + k\, Q^{(k-1)}(t)$$

now we can see that now we can multiply both sides by

$(\beta - t)^k$.

So we get

$$(\beta-t)^k f^{(k)}(t) = \left((t-\beta) Q^{(k)}(t) + k\, Q^{(k-1)}(t) \right) \cdot (\beta -$$

$$= -(\beta - t)^{k+1} \cdot Q^{k}(t) + k\cdot(\beta - t)^{k-1} \cdot Q^{k-1}(t).$$

Now we have $\dfrac{(\beta - t)^k \cdot f(t)}{k!}$ would give us.

$$-\frac{(\beta - t)^{k+1} \cdot Q^{k}(t)}{k!} + \frac{(\beta - t)^{k-1} \cdot Q^{k-1}(t)}{(k-1)!}$$

Now we can take the summation. Now we can see that the summation would be in the same form as the Taylors Theorem and so we can see that the summation is telescoping so we get.

get:

$$\sum_{k=0}^{n-1} \frac{f^{(k)}(t)(B-t)}{k!} = f(B) - \frac{Q^{n-1}(t)}{(n-1)!} \cdot (B-t)^n.$$

Now we can convert the $f(B)$ to $P(B)+Q$ form and we would get that.

$$P(B) = f(B) - \frac{Q^{n-1}(t)}{(n-1)!} \cdot (B-t)^n.$$

$$f(B) = P(B) + \frac{Q^{n-1}(t)}{(n-1)!} \cdot \cancel{(B-t)}^n.$$

Now we can convert the t to α and we get that

$$f(B) = P(B) + \frac{Q^{n-1}(\alpha)}{(n-1)!} \cdot (B-\alpha)^n.$$

So we have show the statement.

Answer: we have proven the statement.

5.5.22.

(a). we need to prove that f has at most one fixed point.

we can use a proof by contradiction to prove this statement.

we know from the mean value theorem that. if a function has two fixed points which are different then there exists a point between the two points such that if we have a, b then we have c such that

$f(a) = a$, $f(b) = b$.

so $b - a = f(b) - f(a)$

By the mean value theorem, we know that

$f(b) - f(a) = f'(c) \cdot (b - a)$.

so this would mean that. $f'(c) = 1$.

so as $f'(c) \neq 1$, then we see that we cannot have two fixed points, so.

we have at most one fixed point.

Answer = we have prove the statement.

(b) we need to show that
$f(t) = t + (1+e^t)^{-1}$ has no fixed point.

$$f(t) = t + \frac{1}{1+e^t}$$

So we need $f(t) = t$.

$$\text{so } f(t) = t = t + \frac{1}{1+e^t}.$$

$$t = t + \frac{1}{1+e^t}$$

$$0 = \frac{1}{1+e^t}$$

Here we need $\frac{1}{1+e^t} = 0$.

we know that we cannot get $\frac{1}{1+e^t}$ to equal 0.

So even though we see that $0 < f'(t) < 1$ for all real t, we can see that we cannot get a fixed point.

Answer = we have proven the statement.

c). we need to show that we have a fixed point ~~a~~ of f when $|f'(t)| \le A$, where $A < 1$.

So we know that $-A \le f'(t) \le A$, so $f'(t)$ is bounded.

So we can see that we can have a sequence given by x_n which is a Cauchy sequence.

For $n > m > N$

So we know that $|x_n - x_m| < \epsilon$.

$$|x_n - x_m| = |x_n - x_{n-1} + x_{n-1} + \cdots - x_m|$$

$$\le |x_n - x_{n-1}| + \cdots + |x_{m+1} - x_m|.$$

So now we can use induction and the mean value theorem to see that

$$\frac{|x_{n+1} - x_n|}{|x_2 - x_1|} \le A^{n-1}$$

so for $|x_n - x_m|$ we have that.

$$\underline{|x_{m+1} - x_m|} + \cdots + |x_n - x_{n-1}|$$

$$\leq A^{m-1}|x_2 - x_1| + \cdots + A^{n-2}|x_2 - x_1|.$$

So we have.

$$|x_n - x_m| \leq A^{m-1}|x_2 - x_1| + \cdots + A^{n-2}|x_2 - x_1|$$

$$\leq |x_2 - x_1| \left(A^{m-1} + \cdots + A^{n-2} \right).$$

$$= |x_2 - x_1| \cdot \sum_{k=m-1}^{n-2} A^k.$$

$$= |x_2 - x_1| \cdot \frac{A^{m-1}}{1-A}$$

So as $n > m$, we know that.

$$\frac{A^{m-1}}{1-A} \leq \frac{A^N}{1-A}.$$

So now we have $|x_n - x_m| \leq \frac{A^N}{1-A} \cdot |x_2 - x_1|$

So we need to show that it is Cauchy when $N \to \infty$.

As $\lim A^N = 0$ as $0 \leq A \leq 1$.

$n \to \infty \qquad i \to \infty$

so now as we have a Cauchy sequence we can now let the limit point be X and now we can prove this.

From the question we know that
$x = \lim_{n \to \infty} x_n$.

so as $X = \lim_{n \to \infty} x_n$, we also know from the question that
$f(x_n) = x_{n+1}$.

So here we can see that.

$$\lim x_{n+1} = \lim f(x_n)$$

$$\lim f(x_n) = f(\lim x_n)$$

$$f(\lim_{n \to \infty} x_n) = f(x).$$

so as $\lim x_n = \lim x_{n+1}$, we can see that

$f(x) = x$.

So we have shon the statement.

Answer = we have proven the ~~statement~~

(d) we need to show that it can be visualized by a zig zag path given by.

$(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3,$

so now we can see that we can start with the point given by $(x_1, f(x_1))$ here x_1 is an arbitrary value. Now for each point after, we can see that the points following won be $(x_n, f(x_n))$ So from each point on the graph, one can move to the line $y = x$, and then to the next point in the chain. So here as we are moving from (x_n, y_n) to (x_{n+1}, y_{n+1}), we see that for each we have a zigzag pattern.

Each $y_n = f(x_n)$, so the graph of $(x, f(x_n))$ to a from $y=x$ would give us the zig zag ~~pattern~~

we have shown this statement

Answer = we have proven this ~~statement~~