Math 104 Homework 1

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Ross 1.10

Theorem: Prove $(2n+1)+(2n+3)+(2n+5)+...+(4n-1) = 3n^2$ for all positive integers n.
Proof:
$P(n)$ is the statement "$(2n+1)+(2n+3)+(2n+5)+...+(4n-1) = 3n^2$"
Note: The left hand side of the above equation sums a sequence that starts at the $(2n+1)$ term and counts up by 2's until the sequence reaches the final $(4n-1)$ term.
Base Case: $P(1)$
The $(2n+1)$ term equals 3 and the $(4n-1)$ term is also 3. We can observe that these are the same term and, thus, the left hand side sums only one element. Substituting into the right hand side we can confirm the base case is true.

$$(2(1)+1) = 3(1)^2 \Rightarrow 3 = 3$$

Induction Step: $P(n) \Rightarrow P(n+1)$
Assume $P(n)$ is true; therefore it is the case that

$$(2n+1)+(2n+3)+...+(4n-1) = 3n^2$$

We can rewrite the left hand side in terms of $n+1$.

$$(2(n+1)-1)+(2(n+1)+1)+...+(4(n+1)-5) = 3n^2$$

We then add $(4(n+1)-3)$ and $(4(n+1)-1)$ and subtract $(2(n+1)-1)$ on both sides and simplify.

$$(2(n+1)+1)+...+(4(n+1)-5)+(4(n+1)-3)+(4(n+1)-1)$$
$$= 3n^2-(2(n+1)-1)+(4(n+1)-3)+(4(n+1)-1)$$

$$\Rightarrow (2(n+1)+1)+...+(4(n+1)-1) = 3n^2-6n+3$$

$$\Rightarrow (2(n+1)+1)+...+(4(n+1)-1) = 3(n+1)^2$$

The above equation is $P(n+1)$, proving $P(n) \Rightarrow P(n+1)$. By the principle of mathematical induction, $P(n)$ holds for all positive integers n.

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Math 104

Ross 1.12

The binomial theorem:

$$(a+b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b + ... + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}a^0b^n$$

for $n \geq 0$ where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

for $n, k \geq 0$.
(a) Let $P(n)$ be the statement

$$(a+b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b + ... + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}a^0b^n$$

P(1)

$$(a+b)^1 = \binom{1}{0}a^1b^0 + \binom{1}{0}a^0b^1$$

$$\Rightarrow a+b = a+b$$

$P(1)$ is true.
P(2)

$$(a+b)^2 = \binom{2}{0}a^2b^0 + \binom{2}{1}a^1b^1 + \binom{2}{2}a^0b^2$$

$$\Rightarrow a^2+2ab+b^2 = a^2+2ab+b^2$$

$P(2)$ is true.
P(3)

$$(a+b)^3 = \binom{3}{0}a^3b^0 + \binom{3}{1}a^2b^1 + \binom{3}{2}a^1b^2 + \binom{3}{3}a^0b^3$$

$$\Rightarrow a^3+3a^2b+3ab^2+b^3 = a^3+3a^2b+3ab^2+b^3$$

$P(3)$ is true.
(b) Theorem:

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

for $k \geq 0$.
Proof:

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}$$

$$\Rightarrow \binom{n}{k} + \binom{n}{k-1} = \frac{n!(n-k+1)}{(k)!(n-k+1)!} + \frac{n!(k)}{(k)!(n-k+1)!}$$

$$\Rightarrow \binom{n}{k} + \binom{n}{k-1} = \frac{n!(n+1)}{(k)!(n-k+1)!}$$

$$\Rightarrow \binom{n}{k} + \binom{n}{k-1} = \frac{(n+1)!}{(k)!((n+1)-k)!}$$

$$\Rightarrow \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

(c) Theorem: $P(n)$ is true for all $n \geq 0$
Base Case: $P(0)$

$$(a+b)^0 = \binom{0}{0} a^0 b^0 \Rightarrow 1 = 1$$

$P(0)$ is true. (Furthermore $P(1)$, $P(2)$, and $P(3)$ were proven in part (a).)
Induction Step: $P(n) \Rightarrow P(n+1)$
Assume $P(n)$ is true; therefore it is the case that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Multiply both sides by $(a+b)$

$$(a+b)(a+b)^n = (a+b) \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$\Rightarrow (a+b)^{n+1} = \sum_{k=0}^{n} \binom{n}{k} a^{n-k+1} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1}$$

Remove the $k = 0$ term from the first summation and the $k = n$ term from the second summation.

$$(a+b)^{n+1} = \binom{n}{0} a^{n+1} b^0 + \sum_{k=1}^{n} \binom{n}{k} a^{n-k+1} b^k + \sum_{k=0}^{n-1} \binom{n}{k} a^{n-k} b^{k+1} + \binom{n}{n} a^0 b^{n+1}$$

Re-index the second summation to start at $k = 1$.

$$(a+b)^{n+1} = \binom{n}{0} a^{n+1} b^0 + \sum_{k=1}^{n} \binom{n}{k} a^{n-k+1} b^k + \sum_{k=1}^{n} \binom{n}{k-1} a^{n-k+1} b^k + \binom{n}{n} a^0 b^{n+1}$$

$$\Rightarrow (a+b)^{n+1} = \binom{n}{0} a^{n+1} b^0 + \sum_{k=1}^{n} \left(\binom{n}{k} a^{n-k+1} b^k + \binom{n}{k-1} a^{n-k+1} b^k \right) + \binom{n}{n} a^0 b^{n+1}$$

Using the result from part (b) we can combine terms in the summation.

$$(a+b)^{n+1} = \binom{n}{0} a^{n+1} b^0 + \sum_{k=1}^{n} \binom{n+1}{k} a^{n-k+1} b^k + \binom{n}{n} a^0 b^{n+1}$$

Using the fact that $\binom{n+1}{0} = \binom{n}{0} = \binom{n+1}{n+1} = \binom{n}{n} = 1$ we can make convenient substitutions.

$$(a+b)^{n+1} = \binom{n+1}{0} a^{n+1} b^0 + \sum_{k=1}^{n} \binom{n+1}{k} a^{(n+1)-k} b^k + \binom{n+1}{n+1} a^0 b^{n+1}$$

We can now reincorporate terms into the sum.

$$(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{(n+1)-k} b^k$$

The above statement is $P(n+1)$, proving that $P(n) \Rightarrow P(n+1)$. By the principle of mathematical induction, P(n) is true for all $n \geq 0$

Ross 2.1

Theorem: $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{24}$, and $\sqrt{31}$ are not rational numbers.
Proof:
$\sqrt{3}$**:**
$\sqrt{3}$ is a zero of $x^2 - 3 = 0$. By the Rational Zeroes Theorem (RZT), the only possible rational zeroes of the above equation are $\pm 1, \pm 3$. Substitution shows that none of these possible zeroes is a zero to the above equation; therefore, the above equation has no rational zeroes. Since $\sqrt{3}$ is a zero of the above equation, it is not rational.
$\sqrt{5}$**:**
$\sqrt{5}$ is a zero of $x^2 - 5 = 0$. The only possible rational zeroes are $\pm 1, \pm 5$. None of these are in fact zeroes; so, there are no rational zeroes. Therefore, $\sqrt{5}$ is not rational.
$\sqrt{7}$**:**
$\sqrt{7}$ is a zero of $x^2 - 7 = 0$. The only possible rational zeroes are $\pm 1, \pm 7$. None of these are in fact zeroes; so, there are no rational zeroes. Therefore, $\sqrt{7}$ is not rational.
$\sqrt{24}$**:**
$\sqrt{24}$ is a zero of $x^2 - 24 = 0$. The only possible rational zeroes are $\pm 1, \pm 2$, $\pm 3, \pm 4$, $\pm 6, \pm 8$, $\pm 12, \pm 24$. None of these are in fact zeroes; so, there are no rational zeroes. Therefore, $\sqrt{24}$ is not rational.
$\sqrt{31}$**:**
$\sqrt{31}$ is a zero of $x^2 - 31 = 0$. The only possible rational zeroes are $\pm 1, \pm 31$. None of these are in fact zeroes; so, there are no rational zeroes. Therefore, $\sqrt{31}$ is not rational.

Ross 2.2

Theorem: $\sqrt[3]{2}$, $\sqrt[7]{5}$, $\sqrt[4]{13}$ are not rational numbers.
Proof:
$\sqrt[3]{2}$**:**
$\sqrt[3]{2}$ is a zero of $x^3 - 2 = 0$. The only possible rational zeroes are $\pm 1, \pm 2$. None of these are in fact zeroes; so, there are no rational zeroes. Therefore, $\sqrt[3]{2}$ is not rational.
$\sqrt[7]{5}$**:**
$\sqrt[7]{5}$ is a zero of $x^7 - 5 = 0$. The only possible rational zeroes are $\pm 1, \pm 5$. None of these are in fact zeroes; so, there are no rational zeroes. Therefore, $\sqrt[7]{5}$ is not rational.
$\sqrt[4]{13}$**:**
$\sqrt[4]{13}$ is a zero of $x^4 - 13 = 0$. The only possible rational zeroes are $\pm 1, \pm 13$. None of these are in fact zeroes; so, there are no rational zeroes. Therefore, $\sqrt[4]{13}$ is not rational.

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Math 104

Ross 2.7

(a) Theorem: $\sqrt{4+2\sqrt{3}}-\sqrt{3}$ is rational.
Proof:

$$x = \sqrt{4+2\sqrt{3}}-\sqrt{3}$$

$$\Rightarrow x+\sqrt{3} = \sqrt{4+2\sqrt{3}}$$

$$\Rightarrow (x+\sqrt{3})^2 = \left(\sqrt{4+2\sqrt{3}}\right)^2 \Rightarrow x^2+2\sqrt{3}x+3 = 4+2\sqrt{3}$$

$$\Rightarrow x^2+2\sqrt{3}x-1-2\sqrt{3} = 0$$

$$\Rightarrow (x-1)(x+1+2\sqrt{3}) = 0$$

$$\Rightarrow x = 1, -1-2\sqrt{3}$$

$x=-1-2\sqrt{3}$ is the extraneous solution. $x>0$ since $\sqrt{4+2\sqrt{3}}>2$ and $\sqrt{3}<2$. Therefore, $x=1$ is the only possible solution. 1 is a rational number so x is rational.
(b) Theorem: $\sqrt{6+4\sqrt{2}}-\sqrt{2}$ is rational.
Proof:

$$y = \sqrt{6+4\sqrt{2}}-\sqrt{2}$$

$$\Rightarrow y+\sqrt{2} = \sqrt{6+4\sqrt{2}}$$

$$\Rightarrow (y+\sqrt{2})^2 = \left(\sqrt{6+4\sqrt{2}}\right)^2 \Rightarrow y^2+2\sqrt{2}y+2 = 6+4\sqrt{2}$$

$$\Rightarrow y^2+2\sqrt{2}y-4-4\sqrt{2} = 0$$

$$\Rightarrow (y-2)(y+2+2\sqrt{2}) = 0$$

$$\Rightarrow y = 2, -2-2\sqrt{2}$$

$y=-2-2\sqrt{2}$ is the extraneous solution. $y>0$ since $\sqrt{6+4\sqrt{2}}>2$ and $\sqrt{2}<2$. Therefore, $y=2$ is the only possible solution. 2 is rational so y is rational.

Ross 3.6

(a) Theorem: $|a+b+c| \leq |a|+|b|+|c|$ for all $a,b,c \in \mathbb{R}$.
Proof:
Consider some $a,b,c \in \mathbb{R}$ and $z \equiv b+c$. It follows that $z \in \mathbb{R}$. According to the triangle inequality:

$$|a+z| \leq |a|+|z|$$

Substituting in $z=b+c$ we get

$$\text{(i)}|a+b+c| \leq |a|+|b+c|$$

Saving inequality (i) for later, we can separately use the triangle inequality to determine that

$$|b + c| \leq |b| + |c|$$

$$\Rightarrow |a| + |b + c| \leq |a| + |b| + |c|$$

Sandwiching this inequality with inequality (i) we conclude that

$$|a + b + c| \leq |a| + |b| + |c|$$

for all $a, b, c \in \mathbb{R}$.

(b) Theorem: $|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$ for n numbers $a_1, a_2, ..., a_n \in \mathbb{R}$.

Proof:

Let $P(n)$ be the statement that "$|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$ for n numbers $a_1, a_2, ..., a_n \in \mathbb{R}$."

Base Cases: $P(1), P(2)$

$n = 1$ is a trivial case since it is necessarily the case that $|a_1| = |a_1| \Rightarrow |a_1| \leq |a_1|$. $n = 2$ is just the case of the Triangle Inequality, which this proof takes to be true.

Induction Step: $P(n) \Rightarrow P(n + 1)$

Assume $P(n)$ is true; therefore, it is the case that

$$|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$$

for n numbers $a_1, a_2, ..., a_n \in \mathbb{R}$. $z \equiv a_1 + a_2 + ... + a_n$. It follows that $z \in \mathbb{R}$. Now consider some $a_{n+1} \in \mathbb{R}$ According to the triangle inequality:

$$|z + a_{n+1}| \leq |z| + |a_{n+1}|$$

Substituting in $z = a_1 + a_2 + ... + a_n$ we get

$$\text{(i)}|a_1 + a_2 + ... + a_n + a_{n+1}| \leq |a_1 + a_2 + ... + a_n| + |a_{n+1}|$$

Saving inequality (i) for later, we separately know from our assumption that $P(n)$ is true that

$$|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$$

$$\Rightarrow |a_1 + a_2 + ... + a_n| + |a_{n+1}| \leq |a_1| + |a_2| + ... + |a_n| + |a_{n+1}|$$

Sandwiching this inequality with inequality (i) we conclude that

$$|a_1 + a_2 + ... + a_n + a_{n+1}| \leq |a_1| + |a_2| + ... + |a_n| + |a_{n+1}|$$

for $n + 1$ numbers $a_1, a_2, ..., a_n, a_{n+1} \in \mathbb{R}$. The above statement is $P(n + 1)$, proving that $P(n) \Rightarrow P(n + 1)$. By the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$.

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Math 104

Ross 4.11

Theorem: For some $a, b \in \mathbb{R}$ where $a < b$, there are infinitely many rationals between a and b.
Proof:
Let $P(n)$ be the claim "for some $a, b \in \mathbb{R}$ where $a < b$, there are n rationals between a and b".
Base Case: $P(1)$
Due to the denseness of $\mathbb{Q}$, there is a rational $r_1 \in \mathbb{Q}$ such that $a < r_1 < b$.
Induction Step: $P(n) \Rightarrow P(n+1)$
Assume $P(n)$ is true; therefore it is the case that there are n rationals $r_1, r_2, r_3, ..., r_n$ between a and b. Without loss of generality, we can take r_n to be the smallest rational. Since $r_n \in \mathbb{Q}$, $r_n \in \mathbb{R}$. Due to the denseness of $\mathbb{Q}$, there is a rational $r_{n+1} \in \mathbb{Q}$ such that $a < r_{n+1} < r_n$. This $r_n + 1$ is a distinct rational from the other n rationals since it is less than the smallest r. It is smaller than every other r and, therefore, cannot be equal to any of them. Since $r_{n+1} < r_n$ and $r_n < b$, $r_{n+1} < b$; therefore, $a < r_{n+1} < b$. There are now $n+1$ rationals $r_1, r_2, r_3, ..., r_n, r_{n+1}$ between a and b. The above statement is $P(n+1)$, proving that $P(n) \Rightarrow P(n+1)$. By the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$. Since $P(n)$ holds for infinitely large $n \geq 1$, there are infinitely many rationals between a and b.

Ross 4.14

Let A and B be nonempty bounded subsets of $\mathbb{R}$ and let $A + B$ be the set of all sums $a + b$ where $a \in A$ and $b \in B$.
(a) Theorem: $sup(A + B) = supA + supB$
Proof:
By the definition of supremum, $supA \geq a$ and $supB \geq b$ where a and b are arbitrary elements from A and B, respectively. These inequalities can be added to find that

$$supA + supB \geq a + b$$

Since a and b were arbitrarily selected, $a+b$ is an arbitrary element of $A+B$ by its definition. Therefore, $supA+supB$ is an upper limit for $A+B$ and greater than or equal to the supremum of $A + B$.

$$supA + supB \geq sup(A + B)$$

Separately, it is true that for any $e > 0$, there exists some element $a \in A$ such that

$$a > supA - e$$

If this were not the case and all $a \leq supA - e$, then $supA - e$ is an upper bound for A that is less than its supremum. This would be a contradiction. Similarly, for that same e, there exists some element $b \in B$ such that

$$b > supB - e$$

Adding these two inequalities together yields.

$$a + b > supA + supB - 2e$$

It is possible to prove that the above statement implies that $sup(A+B) \geq supA + supB$ by way of contradiction. Assume that $sup(A + B) < supA + supB$. This implies that

$$e = \frac{supA + supB - sup(A + B)}{2} > 0$$

Substituting this e into the inequality $a + b > supA + supB - 2e$ yields the statement

$$a + b > sup(A + B)$$

for some element $a+b$. We have reached a contradiction since by the definition of supremum, $sup(A+B) \geq a+b$ for all $a+b$ since $a+b \in (A+B)$. Therefore, $sup(A+B) \geq supA+supB$. Combining this statement with $supA + supB \geq sup(A + B)$, which was proven above, we see that $sup(A + B) = supA + supB$.

(b) Theorem: $inf(A + B) = infA + infB$

Proof:

Definition Time! We will define the operator $'$ to act on a set $C \subseteq \mathbb{R}$ such that set $C' = \{-c|c \in C\}$. We can thus determine the following sets using this operator. $A' = \{-a|a \in A\}$. $B' = \{-b|b \in B\}$. $A'+B' = \{a'+b'|a' \in A', b' \in B'\}$, and finally $(A+B)' = \{-c|c \in A+B\}$. It also follows that $C'' = C$.

It is important to first establish that $(A + B)' = A' + B'$. First consider some arbitrary element $x \in (A + B)'$. Since $x \in (A + B)'$, $-x \in (A + B)$; therefore, $-x = a + b$ for some elements $a \in A, b \in B$. $x = -a-b \Rightarrow x = (-a)+(-b)$. Since $-a \in A'$ and $-b \in B'$ by these sets' definitions, $x \in A'+B'$. Since x is an arbitrary element of $(A+B)', (A+B)' \subseteq A'+B'$. Now consider an arbitrary element $y \in A' + B'$. By this set's definition, $y = a' + b'$ where $a' \in A', b' \in B'$. Multiply both sides of the previous equation by -1 to find that $-y = (-a') + (-b')$. Since $a' \in A', -a' \in A$ and since $b' \in B', -b' \in B$. Consequently, $-y \in A + B$ and $y \in (A + B)'$. Since y was an arbitrary element of $A' + B', A' + B' \subseteq (A + B)'$. Finally,

$$A' + B' \subseteq (A + B) \text{ and } (A + B)' \subseteq A' + B' \Rightarrow (A + B)' = A' + B'$$

Next, we will prove that $infC' = -supC$. By the definition of supremum, $supC \geq c$ if $c \in C$. Multiplying both sides of this inequality by -1 shows that $-supC \leq -c$. Since $-c$ is an arbitrary element of C', $-supC$ is a lower bound of C'. Therefore, $-supC \leq infC'$ by the definition of infimum. Furthermore, the statement $-supC \geq infC'$ can be proven by way of contradiction. Assume that $-supC < infC'$. This implies that there must exist some $c \in C$ such that $infC' > -c$. This is because if $infC' \leq all\ -c$ then $-infC' \geq all\ c$. This means $-infC'$ is an upper bound of C, but $-infC' < supC$ which contradicts the definition of supremum. However, the existence of such a $-c$ also creates a contradiction since if $c \in C, -c \in C'$ making $infC' > -c$ false by definition. By way of contradiction, $-supC \geq infC'$. Finally,

$$-supC \geq infC' \ and\ -supC \leq infC' \Rightarrow infC' = -supC$$

From part (a), we know that

$$sup(A' + B') = supA' + supB'$$

Using the fact that $infC' = -supC$:

$$-inf((A' + B')') = -inf(A'') - inf(B'') \Rightarrow inf((A' + B')') = inf(A'') + inf(B'')$$

Using the fact that $A' + B' = (A + B)'$:

$$inf((A + B)'') = inf(A'') + inf(B'')$$

Using the fact that $C'' = C$:

$$inf(A + B) = inf(A) + inf(B)$$

Ross 7.5

(a) $s_n = \sqrt{n^2 + 1} - n$

$$s_n = \frac{\sqrt{n^2+1} - n}{1} * \frac{\sqrt{n^2+1} + n}{\sqrt{n^2+1} + n}$$

$$\Rightarrow s_n = \frac{n^2 + 1 - n^2}{\sqrt{n^2+1} + n} \Rightarrow s_n = \frac{1}{\sqrt{n^2+1} + n}$$

$$\Rightarrow s_n = \frac{\frac{1}{n}}{\sqrt{\frac{n^2}{n^2} + \frac{1}{n^2}} + \frac{n}{n}} = \frac{\frac{1}{n}}{\sqrt{1 + \frac{1}{n^2}} + 1}$$

$$lim(s_n) = \frac{0}{\sqrt{1+0} + 1} = 0$$

(b) $s_n = \sqrt{n^2 + n} - n$

$$s_n = \frac{\sqrt{n^2+n} - n}{1} * \frac{\sqrt{n^2+n} + n}{\sqrt{n^2+n} + n}$$

$$\Rightarrow s_n = \frac{n^2 + n - n^2}{\sqrt{n^2+n} + n} \Rightarrow s_n = \frac{n}{\sqrt{n^2+n} + n}$$

$$\Rightarrow s_n = \frac{\frac{n}{n}}{\sqrt{\frac{n^2}{n^2} + \frac{1}{n^2}} + \frac{n}{n}} = \frac{1}{\sqrt{1 + \frac{1}{n^2}} + 1}$$

$$lim(s_n) = \frac{1}{\sqrt{1+0} + 1} = \frac{1}{2}$$

(c) $s_n = \sqrt{4n^2 + n} - 2n$

$$s_n = \frac{\sqrt{4n^2+n} - 2n}{1} * \frac{\sqrt{4n^2+n} + 2n}{\sqrt{4n^2+n} + 2n}$$

$$\Rightarrow s_n = \frac{4n^2 + n - 4n^2}{\sqrt{4n^2 + n} + 2n} \Rightarrow s_n = \frac{n}{\sqrt{4n^2 + n} + 2n}$$

$$\Rightarrow s_n = \frac{\frac{n}{n}}{\sqrt{\frac{4n^2}{n^2} + \frac{1}{n^2}} + \frac{2n}{n}} = \frac{1}{\sqrt{4 + \frac{1}{n^2}} + 2}$$

$$lim(s_n) = \frac{1}{\sqrt{4 + 0} + 2} = \frac{1}{4}$$