Math 104 Homework 1

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Ross 1.10

Theorem: Prove $(2n+1)+(2n+3)+(2n+5)+...+(4n-1) = 3n^2$ for all positive integers n.
Proof:
$P(n)$ is the statement "$(2n+1)+(2n+3)+(2n+5)+...+(4n-1) = 3n^2$"
Note: The left hand side of the above equation sums a sequence that starts at the $(2n+1)$ term and counts up by 2's until the sequence reaches the final $(4n-1)$ term.
Base Case: $P(1)$
The $(2n+1)$ term equals 3 and the $(4n-1)$ term is also 3. We can observe that these are the same term and, thus, the left hand side sums only one term. Substitution yields

$$(2(1)+1) = 3(1)^2 \Rightarrow 3 = 3$$

Induction Step: $P(n) \Rightarrow P(n+1)$
Assume $P(n)$ is true; therefore it is the case that

$$(2n+1)+(2n+3)+...+(4n-1) = 3n^2$$

We can rewrite the left hand side in terms of $n+1$.

$$(2(n+1)-1)+(2(n+1)+1)+...+(4(n+1)-5) = 3n^2$$

We then add $(4(n+1)-3)$ and $(4(n+1)-1)$ and subtract $(2(n+1)-1)$ on both sides and simplify.

$$(2(n+1)+1)+...+(4(n+1)-5)+(4(n+1)-3)+(4(n+1)-1)$$
$$= 3n^2-(2(n+1)-1)+(4(n+1)-3)+(4(n+1)-1)$$

$$\Rightarrow (2(n+1)+1)+...+(4(n+1)-1) = 3n^2-6n+3$$

$$\Rightarrow (2(n+1)+1)+...+(4(n+1)-1) = 3(n+1)^2$$

The above equation is $P(n+1)$, proving $P(n) \Rightarrow P(n+1)$. By the principle of mathematical induction, $P(n)$ holds for all positive integers n.

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Ross 1.12

The binomial theorem:

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b + ... + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}a^0 b^n$$

for $n \geq 0$ where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

for $n, k \geq 0$.
(a) Let $P(n)$ be the statement

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b + ... + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}a^0 b^n$$

P(1)

$$(a+b)^1 = \binom{1}{0}a^1 b^0 + \binom{1}{0}a^0 b^1$$

$$\Rightarrow a+b = a+b$$

$P(1)$ is true.
P(2)

$$(a+b)^2 = \binom{2}{0}a^2 b^0 + \binom{2}{1}a^1 b^1 + \binom{2}{2}a^0 b^2$$

$$\Rightarrow a^2 + 2ab + b^2 = a^2 + 2ab + b^2$$

$P(2)$ is true.
P(3)

$$(a+b)^3 = \binom{3}{0}a^3 b^0 + \binom{3}{1}a^2 b^1 + \binom{3}{2}a^1 b^2 + \binom{3}{3}a^0 b^3$$

$$\Rightarrow a^3 + 3a^2 b + 3ab^2 + b^3 = a^3 + 3a^2 b + 3ab^2 + b^3$$

$P(3)$ is true.
(b) Theorem:

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

for $k \geq 0$.
Proof:

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}$$

$$\Rightarrow \binom{n}{k} + \binom{n}{k-1} = \frac{n!(n-k+1)}{(k)!(n-k+1)!} + \frac{n!(k)}{(k)!(n-k+1)!}$$

$$\Rightarrow \binom{n}{k} + \binom{n}{k-1} = \frac{n!(n+1)}{(k)!(n-k+1)!}$$

$$\Rightarrow \binom{n}{k} + \binom{n}{k-1} = \frac{(n+1)!}{(k)!((n+1)-k)!}$$

$$\Rightarrow \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

(c) Theorem: $P(n)$ is true for all $n \geq 0$
Base Case: $P(0)$

$$(a+b)^0 = \binom{0}{0} a^0 b^0 \Rightarrow 1 = 1$$

$P(0)$ is true. (Furthermore $P(1)$, $P(2)$, and $P(3)$ were proven in part (a).)
Induction Step: $P(n) \Rightarrow P(n+1)$
Assume $P(n)$ is true; therefore it is the case that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Multiply both sides by $(a+b)$

$$(a+b)(a+b)^n = (a+b) \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$\Rightarrow (a+b)^{n+1} = \sum_{k=0}^{n} \binom{n}{k} a^{n-k+1} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1}$$

Remove the $k = 0$ term from the first summation and the $k = n$ term from the second summation.

$$(a+b)^{n+1} = \binom{n}{0} a^{n+1} b^0 + \sum_{k=1}^{n} \binom{n}{k} a^{n-k+1} b^k + \sum_{k=0}^{n-1} \binom{n}{k} a^{n-k} b^{k+1} + \binom{n}{n} a^0 b^{n+1}$$

Re-index the second summation to start at $k = 1$.

$$(a+b)^{n+1} = \binom{n}{0} a^{n+1} b^0 + \sum_{k=1}^{n} \binom{n}{k} a^{n-k+1} b^k + \sum_{k=1}^{n} \binom{n}{k-1} a^{n-k+1} b^k + \binom{n}{n} a^0 b^{n+1}$$

$$\Rightarrow (a+b)^{n+1} = \binom{n}{0} a^{n+1} b^0 + \sum_{k=1}^{n} \left(\binom{n}{k} a^{n-k+1} b^k + \binom{n}{k-1} a^{n-k+1} b^k \right) + \binom{n}{n} a^0 b^{n+1}$$

Using the result from part (b) we can combine terms in the summation.

$$(a+b)^{n+1} = \binom{n}{0} a^{n+1} b^0 + \sum_{k=1}^{n} \binom{n+1}{k} a^{n-k+1} b^k + \binom{n}{n} a^0 b^{n+1}$$

Using the fact that $\binom{n+1}{0} = \binom{n}{0} = \binom{n+1}{n+1} = \binom{n}{n} = 1$ we can make convenient substitutions.

$$(a+b)^{n+1} = \binom{n+1}{0} a^{n+1} b^0 + \sum_{k=1}^{n} \binom{n+1}{k} a^{(n+1)-k} b^k + \binom{n+1}{n+1} a^0 b^{n+1}$$

We can now reincorporate terms into the sum.

$$(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{(n+1)-k} b^k$$

The above statement is $P(n+1)$, proving that $P(n) \Rightarrow P(n+1)$. By the principle of mathematical induction, P(n) is true for all $n \geq 0$

Ross 2.1

Theorem: $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{24}$, and $\sqrt{31}$ are not rational numbers.
Proof:
$\sqrt{3}$**:**
$\sqrt{3}$ is a zero of $x^2 - 3 = 0$. By the Rational Zeroes Theorem (RZT), the only possible rational zeroes of the above equation are $\pm 1, \pm 3$. Substitution shows that none of these possible zeroes is a zero to the above equation; therefore, the above equation has no rational zeroes. Since $\sqrt{3}$ is a zero of the above equation, it is not rational.
$\sqrt{5}$**:**
$\sqrt{5}$ is a zero of $x^2 - 5 = 0$. The only possible rational zeroes are $\pm 1, \pm 5$. None of these are in fact zeroes; so, there are no rational zeroes. Therefore, $\sqrt{5}$ is not rational.
$\sqrt{7}$**:**
$\sqrt{7}$ is a zero of $x^2 - 7 = 0$. The only possible rational zeroes are $\pm 1, \pm 7$. None of these are in fact zeroes; so, there are no rational zeroes. Therefore, $\sqrt{7}$ is not rational.
$\sqrt{24}$**:**
$\sqrt{24}$ is a zero of $x^2 - 24 = 0$. The only possible rational zeroes are $\pm 1, \pm 2$, $\pm 3, \pm 4$, $\pm 6, \pm 8$, $\pm 12, \pm 24$. None of these are in fact zeroes; so, there are no rational zeroes. Therefore, $\sqrt{24}$ is not rational.
$\sqrt{31}$**:**
$\sqrt{31}$ is a zero of $x^2 - 31 = 0$. The only possible rational zeroes are $\pm 1, \pm 31$. None of these are in fact zeroes; so, there are no rational zeroes. Therefore, $\sqrt{31}$ is not rational.

Ross 2.2

Theorem: $\sqrt[3]{2}$, $\sqrt[7]{5}$, $\sqrt[4]{13}$ are not rational numbers.
Proof:
$\sqrt[3]{2}$**:**
$\sqrt[3]{2}$ is a zero of $x^3 - 2 = 0$. The only possible rational zeroes are $\pm 1, \pm 2$. None of these are in fact zeroes; so, there are no rational zeroes. Therefore, $\sqrt[3]{2}$ is not rational.
$\sqrt[7]{5}$**:**
$\sqrt[7]{5}$ is a zero of $x^7 - 5 = 0$. The only possible rational zeroes are $\pm 1, \pm 5$. None of these are in fact zeroes; so, there are no rational zeroes. Therefore, $\sqrt[7]{5}$ is not rational.
$\sqrt[4]{13}$**:**
$\sqrt[4]{13}$ is a zero of $x^4 - 13 = 0$. The only possible rational zeroes are $\pm 1, \pm 13$. None of these are in fact zeroes; so, there are no rational zeroes. Therefore, $\sqrt[4]{13}$ is not rational.

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Ross 2.7

(a) Theorem: $\sqrt{4+2\sqrt{3}}-\sqrt{3}$ is rational.
Proof:

$$x = \sqrt{4+2\sqrt{3}}-\sqrt{3}$$

$$\Rightarrow x+\sqrt{3} = \sqrt{4+2\sqrt{3}}$$

$$\Rightarrow (x+\sqrt{3})^2 = \left(\sqrt{4+2\sqrt{3}}\right)^2 \Rightarrow x^2+2\sqrt{3}x+3 = 4+2\sqrt{3}$$

$$\Rightarrow x^2+2\sqrt{3}x-1-2\sqrt{3} = 0$$

$$\Rightarrow (x-1)(x+1+2\sqrt{3}) = 0$$

$$\Rightarrow x = 1, -1-2\sqrt{3}$$

$x = -1-2\sqrt{3}$ is the extraneous solution. $x > 0$ since $\sqrt{4+2\sqrt{3}} > 2$ and $\sqrt{3} < 2$. Therefore, $x = 1$ is the only possible solution. 1 is a rational number so x is rational.
(b) Theorem: $\sqrt{6+4\sqrt{2}}-\sqrt{2}$ is rational.
Proof:

$$y = \sqrt{6+4\sqrt{2}}-\sqrt{2}$$

$$\Rightarrow y+\sqrt{2} = \sqrt{6+4\sqrt{2}}$$

$$\Rightarrow (y+\sqrt{2})^2 = \left(\sqrt{6+4\sqrt{2}}\right)^2 \Rightarrow y^2+2\sqrt{2}y+2 = 6+4\sqrt{2}$$

$$\Rightarrow y^2+2\sqrt{2}y-4-4\sqrt{2} = 0$$

$$\Rightarrow (y-2)(y+2+2\sqrt{2}) = 0$$

$$\Rightarrow y = 2, -2-2\sqrt{2}$$

$y = -2-2\sqrt{2}$ is the extraneous solution. $y > 0$ since $\sqrt{6+4\sqrt{2}} > 2$ and $\sqrt{2} < 2$. Therefore, $y = 2$ is the only possible solution. 2 is rational so y is rational.

Ross 3.6

(a) Theorem: $|a+b+c| \leq |a|+|b|+|c|$ for all $a, b, c \in \mathbb{R}$.
Proof:
Consider some $a, b, c \in \mathbb{R}$ and $z \equiv b+c$. It follows that $z \in \mathbb{R}$. According to the triangle inequality:

$$|a+z| \leq |a|+|z|$$

Substituting in $z = b+c$ we get

$$\text{(i)}|a+b+c| \leq |a|+|b+c|$$

Saving inequality (i) for later, we can separately use the triangle inequality to determine that

$$|b+c| \leq |b| + |c|$$

$$\Rightarrow |a| + |b+c| \leq |a| + |b| + |c|$$

Sandwiching this inequality with inequality (i) we conclude that

$$|a+b+c| \leq |a| + |b| + |c|$$

for all $a, b, c \in \mathbb{R}$.
(b) Theorem: $|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$ for n numbers $a_1, a_2, ..., a_n \in \mathbb{R}$.
Proof:
Let $P(n)$ be the statement that "$|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$ for n numbers $a_1, a_2, ..., a_n \in \mathbb{R}$."
Base Cases: $P(1), P(2)$
$n = 1$ is a trivial case since it is necessarily the case that $|a_1| = |a_1| \Rightarrow |a_1| \leq |a_1|$. $n = 2$ is just the case of the Triangle Inequality, which this proof takes to be true.
Induction Step: $P(n) \Rightarrow P(n+1)$
Assume $P(n)$ is true; therefore, it is the case that

$$|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$$

for n numbers $a_1, a_2, ..., a_n \in \mathbb{R}$. $z \equiv a_1 + a_2 + ... + a_n$. It follows that $z \in \mathbb{R}$. Now consider some $a_{n+1} \in \mathbb{R}$ According to the triangle inequality:

$$|z + a_{n+1}| \leq |z| + |a_{n+1}|$$

Substituting in $z = a_1 + a_2 + ... + a_n$ we get

$$\text{(i)}|a_1 + a_2 + ... + a_n + a_{n+1}| \leq |a_1 + a_2 + ... + a_n| + |a_{n+1}|$$

Saving inequality (i) for later, we separately know from our assumption that $P(n)$ is true that

$$|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$$

$$\Rightarrow |a_1 + a_2 + ... + a_n| + |a_{n+1}| \leq |a_1| + |a_2| + ... + |a_n| + |a_{n+1}|$$

Sandwiching this inequality with inequality (i) we conclude that

$$|a_1 + a_2 + ... + a_n + a_{n+1}| \leq |a_1| + |a_2| + ... + |a_n| + |a_{n+1}|$$

for $n+1$ numbers $a_1, a_2, ..., a_n, a_{n+1} \in \mathbb{R}$. The above statement is $P(n+1)$, proving that $P(n) \Rightarrow P(n+1)$. By the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$.

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Ross 4.11

Theorem: For some $a, b \in \mathbb{R}$ where $a < b$, there are infinitely many rationals between a and b.
Proof:
Let $a, b \in \mathbb{R}$ where $a < b$. Due to the denseness of $\mathbb{Q}$, there is a rational $r_1 \in \mathbb{Q}$ such that $a < r_1 < b$; therefore, $n \geq 1$ where n is the number of rationals between a and b. Now assume for the sake of contradiction that there are finitely many rationals $R = \{r_1, r_2, r_3, ..., r_n\}$ between a and b and $|R| = n$. Since R is a finite set with cardinality $n \geq 1$ it has a $min(R) = m$. Due to the denseness of $\mathbb{Q}$, there is a rational $r \in \mathbb{Q}$ such that $a < r < m < b$. This premise produces a contradiction since $r \in R$, but $r < min(R)$. So, by way of contradiction, there are infinitely many rational between a and b.

Ross 4.14

Let A and B be nonempty bounded subsets of $\mathbb{R}$ and let $A + B$ be the set of all sums $a + b$ where $a \in A$ and $b \in B$.
(a) Theorem: $sup(A + B) = supA + supB$
Proof:
By the definition of supremum, $supA \geq a$ and $supB \geq b$ where a and b are arbitrary elements from A and B, respectively. These inequalities can be added to find that

$$supA + supB \geq a + b$$

Since a and b were arbitrarily selected, $a+b$ is an arbitrary element of $A+B$ by its definition. Therefore, $supA + supB$ is an upper bound for $A + B$ and greater than or equal to the supremum of $A + B$.

$$supA + supB \geq sup(A + B)$$

Separately, it is true that for any $e > 0$, there exists some element $a \in A$ such that

$$a > supA - e$$

If this were not the case and all $a \leq supA - e$, then $supA - e$ is an upper bound for A that is less than its supremum. This would be a contradiction. Similarly, for that same e, there exists some element $b \in B$ such that

$$b > supB - e$$

Adding these two inequalities together yields.

$$a + b > supA + supB - 2e$$

It is possible to prove that the above statement implies that $sup(A + B) \geq supA + supB$ by way of contradiction. Assume that $sup(A + B) < supA + supB$. This implies that

$$e = \frac{supA + supB - sup(A + B)}{2} > 0$$

Substituting this e into the inequality $a+b > supA + supB - 2e$ yields the statement

$$a + b > sup(A + B)$$

for some element $a+b$. We have reached a contradiction since by the definition of supremum, $sup(A+B) \geq a+b$ for all $a+b$ since $a+b \in (A+B)$. Therefore, $sup(A+B) \geq supA+supB$. Combining this statement with $supA + supB \geq sup(A + B)$, which was proven above, we see that $sup(A + B) = supA + supB$.
(b) Theorem: $inf(A + B) = infA + infB$
Proof:
By the definition of infinum, $infA \leq a$ and $infB \leq b$ where a and b are arbitrary elements from A and B, respectively. These inequalities can be added to find that

$$infA + infB \leq a + b$$

Since a and b were arbitrarily selected, $a+b$ is an arbitrary element of $A+B$ by its definition. Therefore, $infA+infB$ is a lower bound for $A+B$ and less than or equal to the infinum of $A+B$.

$$infA + infB \leq inf(A + B)$$

Separately, it is true that for any $e > 0$, there exists some element $a \in A$ such that

$$a < infA + e$$

If this were not the case and all $a \geq infA + e$, then $infA + e$ is a lower bound for A that is greater than its infinum. This would be a contradiction. Similarly, for that same e, there exists some element $b \in B$ such that

$$b < infB + e$$

Adding these two inequalities together yields.

$$a + b < infA + infB + 2e$$

It is possible to prove that the above statement implies that $inf(A+B) \leq infA+infB$ by way of contradiction. Assume that $inf(A + B) > infA + infB$. This implies that

$$e = \frac{inf(A + B) - infA - infB}{2} > 0$$

Substituting this e into the inequality $a + b < infA + infB + 2e$ yields the statement

$$a + b < inf(A + B)$$

for some element $a + b$. We have reached a contradiction since by the definition of infinum, $inf(A+B) \leq a+b$ for all $a+b$ since $a+b \in (A+B)$. Therefore, $inf(A+B) \leq infA+infB$. Combining this statement with $infA + infB \leq inf(A + B)$, which was proven above, we see that $inf(A + B) = infA + infB$.

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Ross 7.5

(a) $s_n = \sqrt{n^2+1} - n$

$$s_n = \frac{\sqrt{n^2+1} - n}{1} * \frac{\sqrt{n^2+1} + n}{\sqrt{n^2+1} + n}$$

$$\Rightarrow s_n = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} \Rightarrow s_n = \frac{1}{\sqrt{n^2+1}+n}$$

$$\Rightarrow s_n = \frac{\frac{1}{n}}{\sqrt{\frac{n^2}{n^2} + \frac{1}{n^2}} + \frac{n}{n}} = \frac{\frac{1}{n}}{\sqrt{1+\frac{1}{n^2}} + 1}$$

$$lim(s_n) = \frac{0}{\sqrt{1+0}+1} = 0$$

(b) $s_n = \sqrt{n^2+n} - n$

$$s_n = \frac{\sqrt{n^2+n} - n}{1} * \frac{\sqrt{n^2+n} + n}{\sqrt{n^2+n} + n}$$

$$\Rightarrow s_n = \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} \Rightarrow s_n = \frac{n}{\sqrt{n^2+n}+n}$$

$$\Rightarrow s_n = \frac{\frac{n}{n}}{\sqrt{\frac{n^2}{n^2} + \frac{1}{n^2}} + \frac{n}{n}} = \frac{1}{\sqrt{1+\frac{1}{n^2}} + 1}$$

$$lim(s_n) = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$$

(c) $s_n = \sqrt{4n^2+n} - 2n$

$$s_n = \frac{\sqrt{4n^2+n} - 2n}{1} * \frac{\sqrt{4n^2+n} + 2n}{\sqrt{4n^2+n} + 2n}$$

$$\Rightarrow s_n = \frac{4n^2+n-4n^2}{\sqrt{4n^2+n}+2n} \Rightarrow s_n = \frac{n}{\sqrt{4n^2+n}+2n}$$

$$\Rightarrow s_n = \frac{\frac{n}{n}}{\sqrt{\frac{4n^2}{n^2} + \frac{1}{n^2}} + \frac{2n}{n}} = \frac{1}{\sqrt{4+\frac{1}{n^2}} + 2}$$

$$lim(s_n) = \frac{1}{\sqrt{4+0}+2} = \frac{1}{4}$$