Math 104 Homework 2

Cameron Shotwell

January 4, 2022

Ross 9.9

Suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$

(a) Theorem: If lim $s_n = +\infty$, then lim $t_n = +\infty$.
Proof: Suppose there exists sequences (s_n) and (t_n) such that there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$. Furthermore, lim $s_n = +\infty$. This implies that for each $M > 0$ there is a number N such that $n > N$ implies $s_n > M$. Define N' such that $N' = max(\{N, N_o\})$. For all $n > N'$, $t_n \geq s_n > M$; therefore, lim $t_n = +\infty$.

(b) Theorem: If lim $t_n = -\infty$, then lim $s_n = -\infty$.
Proof: Suppose there exists sequences (s_n) and (t_n) such that there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$. Furthermore, lim $t_n = -\infty$. This implies that for each $M > 0$ there is a number N such that $n > N$ implies $s_n < M$. Define N' such that $N' = max(\{N, N_o\})$. For all $n > N'$, $s_n \leq t_n < M$; therefore, lim $s_n = -\infty$.

(c) Theorem: If lim s_n and lim t_n exist, then lim $s_n \leq$ lim t_n.
Proof:
First I will prove that for any converging sequence $b(n)$, if $b_n \geq a$ for all but finitely many n, then lim $b_n \geq a$. Let $b(n)$ be a converging sequence such that $b_n < a$ for finitely many n. Let the set of all such n be denoted $A = \{n | b_n < a\} = \{n_1, n_2, ..., n_k\}$ where $k \in \mathbb{N}$. Furthermore, let $N_0 = \max A$. Since (b_n) converges, for all $\epsilon > 0$ there exists a number N such that for all $n > N$, $|b_n - b| < \epsilon$ where $b = \lim b_n$. Let $N \geq N_0$ and assume for the sake of contradiction that $b < a$. This implies that $a - b > 0$. So consider some $\epsilon = a - b > 0$.

$$|b_n - b| < a - b \Rightarrow b_n - b < a - b \Rightarrow b_n < a$$

However, we have reached a contradiction because $n > N_0 \Rightarrow b_n \geq a$. Therefore, by way of contradiction, $b = \lim b_n \geq a$.

Separately, suppose there exists sequences (s_n) and (t_n) such that there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$. Furthermore, suppose that lim s_n and lim t_n exist. Now define $b_n = t_n - s_n$. $s_n \leq t_n$ implies that $b_n = t_n - s_n \geq 0$ for $n > N_0$; therefore, lim $b_n \geq 0$. This implies that lim $t_n -$ lim $s_n \geq 0 \Rightarrow$ lim $t_n \geq$ lim s_n.

Cameron Shotwell
ID: 3035907666
Math 104

Ross 9.15

Theorem: $\lim_{n\to\infty} \frac{a^n}{n!}$ for $a \in \mathbb{R}$.
Proof:
First, I will prove that if all $s_n \neq 0$ and the limit $L = \lim |\frac{s_{n+1}}{s_n}| < 1$, then $\lim s_n = 0$. Assume all $s_n \neq 0$ and the limit $L = \lim |\frac{s_{n+1}}{s_n}| < 1$. Consider some a such that $L < a < 1$. By the definition of limit, for some $\epsilon > 0$, there exists an N_0 that for $n > N_0$,

$$||\frac{s_{n+1}}{s_n}| - L| < \epsilon$$

$$\Rightarrow |\frac{s_{n+1}}{s_n}| - L < \epsilon \Rightarrow |\frac{s_{n+1}}{s_n}| < \epsilon + L$$

Consider some $\epsilon = a - L > 0$.

$$|\frac{s_{n+1}}{s_n}| < a \Rightarrow |s_{n+1}| < a|s_n|$$

Define $N = N_0 + 1$. Next, we want to prove $|s_{N+k}| < a^k|s_N|$ for $k \geq 1$ by induction. For the base case, $|s_{N+1}| < a|s_N|$. For the induction step, assume

$$|s_{N+k}| < a^k|s_N|$$

$$\Rightarrow a|s_{N+k}| < a^{k+1}|s_N| \Rightarrow |s_{N+k+1}| < a|s_{N+k}| < a^{k+1}|s_N|$$
$$\Rightarrow |s_{N+k+1}| < a^{k+1}|s_N|$$

This concludes the induction step, proving $|s_{N+k}| < a^k|s_N|$ for $k \geq 1$. We can rewrite as $|s_n| < a^{n-N}|s_N|$ for $n > N$. The sequence $(b_n)_{n=N}^{\infty}$ where $b_n = a^{n-N}$ converges to 0 since $a < 1$; therefore, the series $|s_N| * (b_n)$ also converges to 0. By the definition of limit, for some $\epsilon' > 0$, there exists an N that for $n > N'$, $|a^{n-N}|s_N| - 0| < \epsilon'$. This implies that for $n > max(\{N', N\})$, $|s_n| < a^{n-N}|s_N| < \epsilon' \Rightarrow |s_n - 0| < \epsilon'$. This means the $\lim s_n = 0$.

Now for the proof you actually came here for. Suppose there is a sequence (t_n) such that $t_n = \frac{a^n}{n!}$ and $a \in \mathbb{R}$. There are now two cases. If $a = 0$, $t_n = 0$ and $\lim t_n = 0$. If $a \neq 0$, then $t_n \neq 0$.

$$L = \lim |\frac{t_{n+1}}{t_n}| = \lim |\frac{\frac{a^{n+1}}{(n+1)!}}{\frac{a^n}{n!}}| = \lim |\frac{a}{n+1}| = 0 < 1$$

Therefore, $\lim t_n = 0$.

Ross 10.7

Theorem: S is a bounded nonempty subset of $\mathbb{R}$ such that sup S is not in S. Prove there is a sequence (s_n) of points in S such that $\lim s_n = \sup S$.
Proof:
Let S be a bounded nonempty subset of $\mathbb{R}$ such that $\sup S \notin S$. Let (t_n) be a sequence such that $t_n = \sup S$. And let (u_n) be the sequence such that $u_n = t_n + \frac{1}{n}$. By the definition of

supremum, for all $\epsilon > 0$, there exists some element $s \in S$ such that $s > \sup S - \epsilon$. Since $1/n > 0$, we can define the sequence (s_n) such that $s_n \in S$ and $s_n > \sup S - \frac{1}{n}$; therefore, $s_n > u_n$ for all n. Furthermore, $\sup S \geq s$ for all $s \in S$. This implies that $t_n \geq s_n > u_n$. Since (t_n) and (u_n) converge to $\sup S$, (s_n) converges to $\sup S$ by the squeeze lemma.

Ross 10.8

Theorem: Let (s_n) be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + ... + s_n)$. (σ_n) is an increasing sequence.
Proof:
Let (s_n) be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + ... + s_n)$.

First, I will prove by induction that $\sigma_n \leq s_n$ for all n. Let $P(n)$ be the statement "$\sigma_n \leq s_n$". For the base case, $P(1)$:

$$\sigma_1 = s_1 \leq s_1$$

For the induction step, $P(n) \Rightarrow P(n+1)$, assume $P(n)$: $\sigma_n \leq s_n$. Since (s_n) is an increasing sequence,

$$\frac{1}{n}(s_1 + s_2 + ... + s_n) \leq s_n \leq s_{n+1}$$
$$\Rightarrow s_1 + s_2 + ... + s_n \leq n s_{n+1}$$
$$\Rightarrow s_1 + s_2 + ... + s_n + s_{n+1} \leq n s_{n+1} + s_{n+1} = (n+1)s_{n+1}$$
$$\Rightarrow \frac{1}{n+1}(s_1 + s_2 + ... + s_n + s_{n+1}) \leq s_{n+1}$$
$$\Rightarrow \sigma_{n+1} \leq s_{n+1}$$

The above statement is $P(n+1)$, proving $P(n) \Rightarrow P(n+1)$. By the principle of mathematical induction, $P(n)$ holds for all positive integers n.

Separately, consider the difference $\sigma_{n+1} - \sigma_n$:

$$\sigma_{n+1} - \sigma_n = \frac{1}{n+1}(s_1 + s_2 + ... + s_n + s_{n+1}) - \frac{1}{n}(s_1 + s_2 + ... + s_n)$$
$$\Rightarrow \sigma_{n+1} - \sigma_n = \frac{s_{n+1}}{n+1} + \frac{n}{n(n+1)}(s_1 + s_2 + ... + s_n) - \frac{n+1}{n(n+1)}(s_1 + s_2 + ... + s_n)$$
$$\Rightarrow \sigma_{n+1} - \sigma_n = \frac{s_{n+1}}{n+1} - \frac{1}{n(n+1)}(s_1 + s_2 + ... + s_n)$$
$$\Rightarrow (n+1)(\sigma_{n+1} - \sigma_n) = s_{n+1} - \frac{1}{n}(s_1 + s_2 + ... + s_n)$$
$$\Rightarrow (n+1)(\sigma_{n+1} - \sigma_n) = s_{n+1} - \sigma_n$$

Using the fact that (s_n) is an increasing sequence, we can say that $s_{n+1} \geq s_n \geq \sigma_n$; therefore $s_{n+1} - \sigma_n \geq 0$ which implies that

$$(n+1)(\sigma_{n+1} - \sigma_n) \geq 0 \Rightarrow \sigma_{n+1} \geq \sigma_n$$

for all n; therefore, (σ_n) is an increasing sequence.

Cameron Shotwell
ID: 3035907666
Math 104

Ross 10.9

Let $s_1 = 1$ and $s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2$ for $n \geq 1$.
(a):

$$s_2 = \left(\frac{1}{2}\right) * 1^2 \Rightarrow s_2 = \frac{1}{2}$$

$$s_3 = \left(\frac{2}{3}\right) * \left(\frac{1}{2}\right)^2 \Rightarrow s_3 = \frac{1}{6}$$

$$s_4 = \left(\frac{3}{4}\right) * \left(\frac{1}{6}\right)^2 \Rightarrow s_4 = \frac{1}{48}$$

(b) Theorem: $\lim s_n$ exists.
Proof:
$s_n^2 \geq 0$ and $\frac{n}{n+1} \geq 0$ implies that $s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2 \geq 0$. This in combination with the fact that $s_1 = 1 \geq 0$ implies that (s_n) is bounded below by 0.

Next I will prove by induction that $s_n \leq 1$ for all n. Let $P(n)$ be the statement that $s_n \leq 1$. For the base case, $P(1)$,

$$s_1 = 1 \leq 1$$

For the induction step, $P(n) \Rightarrow P(n+1)$, Assume $P(n)$. $0 \leq s_n \leq 1 \Rightarrow s_n^2 \leq 1$. Furthermore, $n+1 > n \Rightarrow 0 \leq \frac{n}{n+1} < 1$; therefore, $s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2 \leq 1$. The previous statement is $P(n+1)$, proving $P(n) \Rightarrow P(n+1)$. By the principle of mathematical induction, $P(n)$ holds for all positive integers n.

Separately, $s_n \leq 1$ implies 1 is an upper bound for (s_n). Also, $0 \leq s_n \leq 1$ implies that $s_n^2 \leq s_n$ and since $\frac{n}{n+1} < 1$, $s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2 \leq s_n$; therefore, (s_n) is a monotonically decreasing sequence. Since (s_n) is monotic and bounded, $\lim s_n$ exists.

(c) Theorem: $s = \lim s_n = 0$
Proof: Taking the limit of s_n using its recursive definition, we get

$$\lim s_{n+1} = \lim \left(\left(\frac{n}{n+1}\right) s_n^2\right)$$

$$\lim s_{n+1} = \lim \left(\frac{n}{n+1}\right) \lim s_n^2$$

$$s = s^2 \Rightarrow s = 0, 1$$

Since (s_n) is a monotonically decreasing sequence with $s_2 = \frac{1}{2}$, $s \neq 1 \Rightarrow s = 0$.

Cameron Shotwell
ID: 3035907666
Math 104

Ross 10.10

Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

(a)

$$s_2 = \frac{1}{3}(1+1) \Rightarrow s_2 = \frac{2}{3}$$
$$s_3 = \frac{1}{3}\left(\frac{2}{3}+1\right) \Rightarrow s_3 = \frac{5}{9}$$
$$s_4 = \frac{1}{3}\left(\frac{5}{9}+1\right) \Rightarrow s_4 = \frac{14}{27}$$

(b) Theorem: $s_n > \frac{1}{2}$ for all n.
Proof:
Let $P(n)$ be the statement "$s_n > \frac{1}{2}$".
Base Case: $P(1)$

$$s_1 = 1 > \frac{1}{2}$$

Induction Step: $P(n) \Rightarrow P(n+1)$
Assume $P(n)$.

$$s_n > \frac{1}{2} \Rightarrow s_n + 1 > \frac{3}{2}$$
$$\Rightarrow s_{n+1} = \frac{1}{3}(s_n + 1) > \frac{1}{2}$$

The above statement is $P(n+1)$, proving $P(n) \Rightarrow P(n+1)$. By the principle of mathematical induction, $P(n)$ holds for all positive integers n.

(c) Theorem: (s_n) is a monotonically decreasing sequence.
Proof:
First, I will prove by induction that $s_n = \frac{3^{n-1}+1}{2*3^{n-1}}$ for all n. Let $P(n)$ be the statement "$s_n = \frac{3^{n-1}+1}{2*3^{n-1}}$". For the base case, $P(1)$,

$$s_1 = \frac{3^{1-1}+1}{2*3^{1-1}} = \frac{1+1}{2*1} = 1$$

For the induction step, $P(n) \Rightarrow P(n+1)$, assume $P(n)$.

$$s_n = \frac{3^{n-1}+1}{2*3^{n-1}} \Rightarrow s_n + 1 = \frac{3^{n-1}+1}{2*3^{n-1}} + 1 = \frac{3^{n-1}+1+2*3^{n-1}}{2*3^{n-1}} = \frac{3^n+1}{2*3^{n-1}}$$
$$s_{n+1} = \frac{1}{3}(s_n+1) = \frac{1}{3} * \frac{3^n+1}{2*3^{n-1}} = \frac{3^n+1}{2*3^n}$$

The above statement is $P(n+1)$, proving $P(n) \Rightarrow P(n+1)$. By the principle of mathematical induction, $P(n)$ holds for all positive integers n.

Separately, consider s_{n+1}.

$$s_{n+1} = \frac{3^n+1}{2*3^n} = \frac{3^n}{2*3^n} + \frac{1}{2*3^n} \leq \frac{3^n}{2*3^n} + \frac{1}{2*3^{n-1}}$$

$$\Rightarrow s_{n+1} \leq \frac{3^n}{2*3^n} + \frac{1}{2*3^{n-1}} = \frac{3^{n-1}}{2*3^{n-1}} + \frac{1}{2*3^{n-1}} = \frac{3^{n-1}+1}{2*3^{n-1}} = s_n$$

$s_{n+1} \leq s_n$ for all n implies (s_n) is a monotonically decreasing sequence.

(d) Theorem: $\lim s_n$ exists and $s = \lim s_n = \frac{1}{2}$.
Proof:
Since (s_n) is monotonically decreasing, it is bounded above by $s_1 = 1$. From (b), (s_n) is bounded below by $\frac{1}{2}$. Since (s_n) is monotonic and bounded, $\lim s_n$ exists. Taking the limit of the recursive definition of s_{n+1},

$$\lim s_{n+1} = \lim \frac{1}{3}(s_n+1) \Rightarrow s = \frac{1}{3}(s+1) \Rightarrow s = \frac{1}{2}$$

Ross 10.11

Let $t_1 = 1$ and $t_{n+1} = \left(1 - \frac{1}{4n^2}\right) t_n$

(a) Theorem: $\lim t_n$ exists.
First, I'll prove the claim that (t_n) is bounded below by 0 by induction. Let $P(n)$ be the statement "$t_n > 0$". For the base case, $P(1)$:

$$t_1 = 1 > 0$$

For the induction step, $P(n) \Rightarrow P(n+1)$, assume $P(n)$. Using the fact that $n \geq 1$.

$$4n^2 > 1 \Rightarrow \frac{1}{4n^2} < 1 \Rightarrow 1 - \frac{1}{4n^2} > 0$$

$t_n > 0$ implies that $t_{n+1} = \left(1 - \frac{1}{4n^2}\right) t_n > 0$. The previous statement is $P(n+1)$, proving $P(n) \Rightarrow P(n+1)$. By the principle of mathematical induction, $P(n)$ holds for all positive integers n. Separately, $\frac{1}{4n^2} > 0$ and $t_n > 0$ implies $-\frac{1}{4n^2} * t_n < 0$. Adding t_n to both sides of the previous inequality reveals

$$t_{n+1} = \left(1 - \frac{1}{4n^2}\right) t_n = t_n - \frac{1}{4n^2} * t_n < t_n$$

$t_{n+1} < t_n$ implies (t_n) is a monotonically decreasing sequence. This further implies that $t_1 = 1$ is an upper bound for (t_n). Since (t_n) is monotonic and bounded, $\lim t_n$ exists.

(b) Since (t_n) is monotonically decreasing and bounded below by 0, I am going to guess that $\lim t_n = 0$. It turns out the limit is $\frac{2}{\pi}$. I was definitely off. What an L.

Cameron Shotwell
ID: 3035907666
Math 104

Squeeze Lemma:

Theorem: Let a_n, b_n, c_n be three sequences such that $a_n \leq b_n \leq c_n$ and $L = \lim\, a_n = \lim\, c_n$. $\lim\, b_n = L$.
Proof:
Let a_n, b_n, c_n be three sequences such that $a_n \leq b_n \leq c_n$ and $L = \lim\, a_n = \lim\, c_n$. $L = \lim\, a_n$ implies that for any $\epsilon > 0$, there exists some N_a such that for $n > N_a$, $|a_n - L| < \epsilon$. This implies that $-\epsilon < a_n - L \Rightarrow -\epsilon + L < a_n$. Similarly, for any $\epsilon > 0$, there exists some N_c such that for $n > N_c$, $|c_n - L| < \epsilon$. This implies that $c_n - L < \epsilon \Rightarrow c_n < \epsilon + L$. Now consider $n > N_b = \max\,(\{N_a, N_c\})$. $a_n \leq b_n \leq c_n$ implies

$$-\epsilon + L < a_n \leq b_n \leq c_n < \epsilon + L$$

$$-\epsilon + L < b_n < \epsilon + L$$

$$-\epsilon < b_n - L < \epsilon$$

$$|b_n - L| < \epsilon \text{ for } n > N_b$$

Therefore, $\lim\, b_n = L$.