10.6 (a) Suppose (s_n) is a sequence s.t. $|s_{n+1} - s_n| < 2^{-n}$ $\forall n \in \mathbb{N}$.

Thrm: (s_n) is a cauchy sequence.

pf: Consider some $m, n \in \mathbb{N}$. Assume WLOG $m \geq n$.

WTS that the proposition:

$$|s_m - s_n| < \sum_{i=n}^{m-1} \frac{1}{2^i} = \frac{2^{m-n} - 1}{2^{m-1}} \quad \text{for } m > n$$

Base Step: $m = n+1$:

$$|s_{n+1} - s_n| < \frac{1}{2^n} = \sum_{i=n}^{n} \frac{1}{2^i} = \frac{2^{\cancel{m}-n} - 1}{2^{\cancel{m}\,n-1}}$$

Induction Step: Assume the proposition for m.

$$|s_m - s_n| < \sum_{i=n}^{m-1} \frac{1}{2^i} = \frac{2^{m-n} - 1}{2^{m-1}}$$

$\triangle$ inequality:

$$|s_{m+1} - s_n| = |(s_{m+1} - s_m) + (s_m - s_n)| \leq |s_{m+1} - s_m| + |s_m - s_n|$$

$$< \sum_{i=n}^{m-1} \frac{1}{2^i} + \frac{1}{2^m} = \sum_{i=n}^{(m+1)-1} \frac{1}{2^i} = \frac{2^{m-n} - 1}{2^{m-1}} + \frac{1}{2^m} = \frac{2^{(m+1)-n} - 2}{2^m} + \frac{1}{2^m}$$

$$= \frac{2^{(m+1)-n} - 1}{2^{(m+1)-1}}$$ Thus the proposition is true for $m > n$.

Now, WTS that $|s_m - s_n| < \frac{1}{2^{n-1}}$. The case $m = n$ is trivial since $1/2^{n+1} > 0$. The case $m > n$ uses the fact that

$$|s_m - s_n| < \frac{2^{m-n} - 1}{2^{m-1}} = \frac{2^{m-n}}{2^{m-1}} - \frac{1}{2^{m-1}} = \frac{1}{2^{n-1}} - \frac{1}{2^{m-1}} < \frac{1}{2^{n-1}}.$$

$\therefore$ $|s_m - s_n| < \frac{1}{2^{n-1}}$ $\forall m, n \in \mathbb{N}$. $\lim \frac{1}{2^{n-1}} = 0$

$\to$ for any $\epsilon > 0$, $\exists N$ s.t. $\left|\frac{1}{2^{n-1}} - 0\right| = \frac{1}{2^{n-1}} < \epsilon$.

Select $m, n > N$ and $|s_m - s_n| < \epsilon$. $\to$ (s_n) is cauchy.

$\square$

(b) Suppose instead that $|s_{n+1} - s_n| < \frac{1}{n}$. The result in (a) is not necessarily true.

Similarly to the argument in (a),

$$|s_{n+1} - s_n| < \frac{1}{n} \rightarrow |s_m - s_n| < \sum_{i=n}^{m-1} \frac{1}{i} \quad \text{for } m > n.$$

We cannot use the argument that this sum is bounded since $\sum_{1}^{\infty} \frac{1}{n}$ is a divergent series. For some arbitrarily small $\epsilon > 0$, for any n, we can pick a large enough m s.t. $\sum_{i=n}^{m-1} \frac{1}{i} > \epsilon$ since the series, and every tail of it, diverges.

$a_n = (-1)^n$, $b_n = \frac{1}{n}$, $c_n = n^2$, $d_n = \frac{6n+4}{7n-3}$

11.2 (a) $a_n \to 1, 1, 1, 1, \ldots$ subsequence of all 1's is monotone.
For b_n, c_n, d_n, they are already monotone so any subsequence will be monotone.
$b_n \to b_1, b_3, b_5, b_7 \ldots$ (odd n's)
$c_n \to$ ~~b_2~~ $c_2, c_3, c_4, c_5, \ldots$ (exclude c_1)
$d_n \to d_1, d_3, d_4, d_5, \ldots$ (exclude d_2)

(b) $a_n \to \{1, -1\}$
$b_n \to \{0\}$ since b_n converges to 0
$c_n \to \{+\infty\}$ since c_n diverges to $+\infty$
$d_n \to \{6/7\}$ since d_n converges to $6/7$.

(c) limsup = sup of set of subsequential limits
liminf = inf of set of subsequential limits
$a_n \to \limsup a_n = 1$, $\liminf a_n = -1$
$b_n \to \limsup b_n = \liminf b_n = 0$
$c_n \to \limsup c_n = \liminf c_n = +\infty$
$d_n \to \limsup d_n = \liminf d_n = 6/7$.

(d) b_n & d_n converge because the set of subsequential limits has 1 $\mathbb{R}$ ~~term~~ element. c_n diverges ~~be~~ because the set of subsequential limits contains only $+\infty$. a_n oscillates.

(e) a_n is bounded by -1 & 1.
b_n is bounded by 0 & 1.
c_n is bounded below by 1, but not above.
d_n is bounded ~~above~~ ~~by~~ by $\frac{5}{2}$ & $\frac{6}{7}$.

$s_n = \cos\left(\frac{n\pi}{3}\right)$, $t_n = \frac{3}{4n+1}$, $u_n = \left(-\frac{1}{2}\right)^n$, $v_n = (-1)^n + \frac{1}{n}$

11.3 (a) $s_n \to s_1, s_7, s_{13}, s_{19}$ (since s_{6k+1} for $k \in \mathbb{N}$ equals $1/2$).

$t_n \to t_1, t_3, t_4, t_5, t_6 \ldots$ (just remove t_2 since (t_n) is already monotone).

$u_n \to u_1, u_3, u_5, u_7$ (odd n's means only ~~positive~~ ~~negative~~ negative increasing ~~decreasing~~ terms).

$v_n \to v_1, v_3, v_5, v_7, \ldots$ (~~even~~ odd n's means negative increasing terms).

(b) $s_n \to \{1/2, -1/2, -1, 1\}$

$t_n \to \{0\}$, $u_n = \{0\}$, $v_n = \{-1, 1\}$

(c) $\limsup s_n = 1$, $\liminf s_n = -1$

$\limsup t_n = \liminf t_n = 0$

$\limsup u_n = \liminf u_n = 0$

$\limsup v_n = 1$, $\liminf v_n = -1$.

(d) t_n & u_n converge. v_n & s_n oscillate.

(e) s_n is bounded by -1 & 1

t_n is bounded by $3/5$ & 0.

u_n is bounded by $1/4$ & $-1/2$.

v_n is bounded by $3/2$ & -1.

11.5 (a) Let (q_n) be an enumeration of all rationals in the interval $(0,1]$.

The set of subsequential limits is

$$\{r \mid r \in \mathbb{R} \text{ and } 0 \le r \le 1\}$$

Consider an arbitrarily small $\epsilon > 0$. Due to the denseness of $\mathbb{Q}$, $\therefore$ infinitely many rationals $q : r - \epsilon < q < r$ and infinitely many q s.t. $r < q < r + \epsilon$. In the special cases of $r = 0, 1$, ~~only the [illegible] [illegible]~~ $\exists$ infinitely many rationals q only on the 2nd & 1st intervals, respectively. However this is sufficient to argue that the set $\{n \mid |q_n - r| < \epsilon\}$ is infinite for all $\epsilon > 0$. Now consider a limit outside of the interval $0 \le r \le 1$. If $r < 0$, the set $\{n \mid |q_n - r| < \epsilon\}$ is empty for $\epsilon < |r|$. For $r > 1$, the set $\{n \mid |q_n - r| < \epsilon\}$ is empty for $\epsilon < r - 1$. Therefore the previously described set of subsequential limits is complete.

(b) $\limsup (q_n) = \sup \{r \mid r \in \mathbb{R} \text{ and } 0 \le r \le 1\}$

$\rightarrow \limsup (q_n) = 1$

$\liminf (q_n) = \inf \{r \mid r \in \mathbb{R} \ \& \ 0 \le r \le 1\}$

$\rightarrow \liminf (q_n) = 0$

2. $\limsup s_n = \lim_{N \to \infty} \sup \{s_n : n > N\}$

limsup describes the end behavior of a sequence and provides an upper bound for terms in the tail of a sequence. The "sup" portion of limsup is that limsup acts as an upper bound on terms of a sequence. The "lim" portion of limsup describes how we are looking at the behavior of a tail that starts an infinite number of terms away from the start of the sequence.

limsup applies to sequences in contrast to sup which applies to sets. Furthermore limsup is not an upper bound for the entirety of a sequence. In fact limsup can easily be less than such an upperbound. So the statement limsup(s_n) is larger than every s_n is not strictly true since limsup describes end behavior of a tail, not all of a sequence.