1.10 Thrm: $(2n+1) + (2n+3) \cdots + (4n-1) = 3n^2 \ \forall n \in \mathbb{Z}_+$

Proof: Base case: $n=1$

Note: we start our ~~sum~~ series at the $(2n+1)$ term and add 2 until we arrive at the $(4n-1)$ term.

$(2n+1)$ term $\to 3$
$(4n-1)$ term $\to 3$

Note these are the same term so the left hand side has 1 term: 3. and $3 = 3n^2$ for $n=1$.

Induction step:

- Assume the theorem holds for a given n:

$\to (2n+1) + (2n+3) \cdots + (4n-1) = 3n^2$

- Rewrite left hand side in terms of $n+1$:

$\to (2(n+1)-1) + (2(n+1)+1) \cdots + (4(n+1)-5) = 3n^2$

- add $(4(n+1)-3)$ & $(4(n+1)-1)$ to both sides & subtract $(2(n+1)-1)$ from both sides to get:

$\to (2(n+1)+1) \cdots (4(n+1)-5) + (4(n+1)-3) + (4(n+1)-1)$
$= 3n^2 + 8(n+1) - 3 - 1 - 2(n+1) + 1$

so $(2(n+1)+1) \cdots + (4(n+1)-1) = 3n^2 - 6n + 6 - 3$
$= 3n^2 - 6n + 3$

$(2(n+1)+1) \cdots + (4(n+1)-1) = 3(n+1)^2$

Thus we have proven that $P(n) \to P(n+1)$. By the principle of mathematical induction, $P(n)$ holds for all n. $\blacksquare$

1.11 <u>Thrm</u>: For each $n \in \mathbb{N}$, $P(n)$ is the assertion that "n^2+5n+1 is an even integer."

(a) <u>Prove $P(n) \to P(n+1)$</u>:

Assume $P(n) \to 2 \mid n^2+5n+1$

$\therefore$ $2k = n^2+5n+1$ where $k \in \mathbb{Z}$

now consider $P(n+1)$

$$(n+1)^2 + (n+1)\cdot 5 + 1 = n^2+2n+1 + 5n+5+1$$
$$= n^2+7n+7$$
$$= (n^2+5n+1) + 2n+6$$

substitute to get

$$(n+1)^2+5(n+1)+1 = 2k+2n+6$$
$$= 2(k+n+6)$$

since $k, n \in \mathbb{Z}$, $2 \mid 2(k+n+6)$

$\therefore$ $2 \mid (n+1)^2+5(n+1)+1 = P(n+1)$

so $P(n) \to P(n+1)$ $\boxed{//}$

(b) $P(n)$ is false for all $n \in \mathbb{N}$.

<u>Proof by cases</u>:

<u>case 1</u>: n is even $\to n = 2k$ where $k \in \mathbb{Z}$

$$\therefore\ n^2+5n+1 = (2k)^2+5(2k)+1$$
$$= 4k^2+10k+1$$
$$n^2+5n+1 = 2(2k^2+5k)+1$$

n^2+5n+1 is of the form $2m+1$ where $m = 2k^2+5k \in \mathbb{Z}$

so n^2+5n+1 is odd and $P(n)$ is false.

<u>case 2</u>: n is odd $\to n = 2k+1$ where $k \in \mathbb{Z}$

$$n^2+5n+1 = (2k+1)^2+5(2k+1)+1$$
$$= 4k^2+4k+1+10k+5+1$$
$$n^2+5n+1 = 2(2k^2+7k+3)+1$$

n^2+5n+1 is of the form $2m+1$ where $m = 2k^2+7k+3 \in \mathbb{Z}$

so n^2+5n+1 is odd $\to P(n)$ is false.

since n ~~must~~ be even or odd, $P(n)$ is false for all n $\boxed{//}$

<u>moral</u>: induction means nothing without a <u>base case</u>

2.1 Prove $\sqrt{3}$ is irrational.

$\sqrt{3}$ is a solution to $x^2-3=0$

By the rational zeroes theorem (RZT), the only possible rational solutions to $x^2-3=0$ are of the form p/q where $p, q \in \mathbb{Z}$ and $p|-3$ while $q|1$.

-> The only rational solutions are $\pm 1, \pm 3$ and substitution shows that no possible rational solution is a solution to $x^2-3=0$. $\therefore$ $x^2-3=0$ has no rational solutions -> $\sqrt{3}$, a solution to $x^2-3=0$, must be irrational $\boxed{\text{N}}$

Prove $\sqrt{5}$ is irrational

$\sqrt{5}$ is a solution to $x^2-5=0$.

By the RZT, the only possible rational solutions to $x^2-5=0$ are $\pm 1, \pm 5$. Substitution shows that none of the possible rational zeroes are in fact zeroes -> $x^2-5=0$ has no rational solution -> $\sqrt{5}$ is irrational. $\boxed{\text{N}}$

Prove $\sqrt{7}$ is irrational

$\sqrt{7}$ solves $x^2-7=0$. $\Leftarrow f(x)=0$

By the RZT, the only possible rational zeroes of $f(x)=0$ are $\pm 7, \pm 1$. Substitution shows these possibilities are not solutions -> $f(x)=0$ has no rational solutions -> $\sqrt{7}$ is irrational $\boxed{\text{N}}$

Prove $\sqrt{24}$ is irrational

$\sqrt{24}$ solves $x^2-24=0$ -> $f(x)=0$

By the RZT, the possible rational zeroes of $f(x)=0$ are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$. Substitution shows none of these possible rational zeroes solve $f(x)=0$ so $f(x)=0$ has no rational solution -> $\sqrt{24}$ is irrational. $\boxed{\text{N}}$

Prove $\sqrt{31}$ is irrational

$\sqrt{31}$ solves $f(x)=x^2-31=0$. By the RZT, the possible rational zeroes of $f(x)=0$ are $\pm 1, \pm 31$. Substitution shows these are not solutions to $f(x)=0$, so $f(x)=0$ has no rational solutions -> $\sqrt{31}$ is irrational

2.2 $\sqrt[3]{2}$ is irrational

$\sqrt[3]{2}$ is a solution to $f(x) = x^3 - 2 = 0$

By the Rational Zeroes Theorem (RZT), the only possible rational zeroes of $f(x)=0$ are $\pm 1, \pm 2$. Substitution shows none of these are solutions so $f(x)=0$ has no rational zeroes. $\to$ $\sqrt[3]{2}$ is irrational since it is a zeroe of $f(x)=0$ ☑

$\sqrt[7]{5}$ is irrational

$\sqrt[7]{5}$ is a solution to $x^7 - 5 = 0 = f(x)$

By the RZT, the only possible rational zeroes of $f(x)=0$ are $\pm 1, \pm 5$. Substitution shows that none of these are solutions to $f(x)=0$. $f(x)=0$ has no rational zeroes $\to$ $\sqrt[7]{5}$ is irrational. ☑

$\sqrt[4]{13}$ is irrational

$\sqrt[4]{13}$ solves $x^4 - 13 = 0 = f(x)$

By the RZT, the only possible rational zeroes of $f(x)=0$ are $\pm 1, \pm 13$. Substitution shows that none of these are solutions to $f(x)=0$. $f(x)=0$ has no rational zeroes $\to$ $\sqrt[4]{13}$ must be irrational.

(a)

2.7 $n = \sqrt{4+2\sqrt{3}} - \sqrt{3} \rightarrow n + \sqrt{3} = \sqrt{4+2\sqrt{3}}$

$n^2 + 2\sqrt{3}n + 3 = 4 + 2\sqrt{3}$

$n^2 + 2\sqrt{3}n - 1 - 2\sqrt{3} = 0$

$(n-1)(n+1+2\sqrt{3}) = 0$

$n = 1, -2\sqrt{3} - 1$

$-2\sqrt{3}-1$ is the extraneous solution since

$\sqrt{4+2\sqrt{3}} - \sqrt{3} > 0$ since $\sqrt{4+2\sqrt{3}} > 2$ & $\sqrt{3} < 2$.

$\therefore \boxed{n = 1}$ and $1 \in \mathbb{Q} \rightarrow n \in \mathbb{Q}$ $\blacksquare$

(b) $n = \sqrt{6+4\sqrt{2}} - \sqrt{2}$

$n + \sqrt{2} = \sqrt{6+4\sqrt{2}}$

$n^2 + 2\sqrt{2}n + 2 = 6 + 4\sqrt{2}$

$n^2 + 2\sqrt{2}n - 4 - 4\sqrt{2} = 0$

$(n-2)(n+2+2\sqrt{2}) = 0$

$n = 2, -2-2\sqrt{2}$

$-2-\sqrt{2}$ is the extraneous solution ~~since~~ since

$\sqrt{6+4\sqrt{2}} - \sqrt{2} > 0$ because $\sqrt{6+4\sqrt{2}} > 2$ & $\sqrt{2} < 2$ so

$\boxed{n = 2 \text{ \& } 2 \in \mathbb{Q}} \rightarrow n \in \mathbb{Q}$ $\blacksquare$

Theorem 3.1 (i) $\underline{a+c=b+c \to a=b}$

$a+c=b+c$

add $(-c)$ to both sides to get

$a+c+(-c)=b+c+(-c)$

A1 $\to$ $a+(c+(-c)) = b+(c+(-c))$

A4 $\to$ $a+0 = b+0$

A3 $\to$ $a=b$ $\boxed{\checkmark}$

(ii) $\underline{a\cdot 0 = 0 \text{ for all } a}$

$a\cdot 0 = a\cdot(0)$

$a\cdot 0 = a\cdot[0+0]$

DL $\to$ $a\cdot 0 = a\cdot 0 + a\cdot 0$

A3 $\to$ $a\cdot 0 + 0 = a\cdot 0 + a\cdot 0$

A2 $\to$ $0 + a\cdot 0 = a\cdot 0 + a\cdot 0$

3.1 (i) $\to$ $0 = a\cdot 0$ $\boxed{\checkmark}$

(iii) $\underline{(-a)b = -ab \text{ for all } a, b}$

3.1(ii) $\to 0 = 0\cdot b$

A4 $\to$ $0 = (a+(-a))\cdot b$

DL $\to$ $0 = ab + (-a)b$

separately, A4 $\to$ $ab + (-ab) = 0$ so

$ab + (-ab) = ab + (-a)b$

A2, Thrm 3.1 (i) $\to$ $-ab = (-a)b$

(iv) $\underline{(-a)(-b) = ab}$

3.1 (ii) $0 = 0\cdot(-b)$

A4 $\to$ $0 = (a+(-a))(-b)$

DL $\to$ $0 = a(-b) + (-a)(-b)$

M2 $\to$ $0 = (-b)(a) + (-a)(-b)$

3.1 (iii) $0 = -ba + (-a)(-b)$

M2 $\to$ $0 = -ab + (-a)(-b)$

separately, A4 $\to$ $ab + (-ab) = 0$, A2 $\to$ $-ab + ab = 0$

combine $\to$ $-ab + ab = -ab + (-a)(-b)$

3.1 (i) $\to$ $ab = (-a)(-b)$ $\boxed{\checkmark}$

Thrm 3.1 (v) $ac = bc$ ∃ $c \neq 0 \rightarrow a = b$

$ac = bc$

M4 → since $c \neq 0$, ∃ an element c^{-1} s.t. $c \cdot c^{-1} = 1$

$(ac)(c^{-1}) = (bc)(c^{-1})$

M1 → $a(cc^{-1}) = b(cc^{-1})$

M4 → $a(1) = b(1)$

M3 → $a = b$ $\boxed{\checkmark}$

(vi) $ab = 0 \rightarrow a = 0$ or $b = 0$

~~WLOG, we can assume $b \neq 0$~~

Two cases:

case 1: $b = 0$ Ta-da!

case 2: $b \neq 0$

Thrm 3.1 (ii) → $b \cdot 0 = 0$

M2 → $0 = 0 \cdot b$

Substitute $ab = 0 \rightarrow ab = 0 \cdot b$

Thrm 3.1 (v) → $a = 0$ since $b \neq 0$. $\boxed{\checkmark}$

Thrm 3.2

(i) <u>If $a \leq b$, then $-b \leq -a$</u>;

$a \leq b$

O4 $\rightarrow$ $a + (-a) \leq b + (-a)$

O4 $\rightarrow$ $a + (-a) + (-b) \leq b + (-a) + (-b)$

A1, A2 $\rightarrow$ $(a + (-a)) + (-b) \leq (-a) + (b + (-b))$

A4, $\rightarrow$ $0 + -b \leq -a + 0$

A3, A2 $\rightarrow$ $-b \leq -a$ $\blacksquare$

(ii) <u>If $a \leq b$ & $c \leq 0$ then $bc \leq ac$</u>

$a \leq b$ & $c \leq 0$

3.2 (i) $\rightarrow$ $-c \geq 0$

O5 $\rightarrow$ $a(-c) \leq b(-c)$

3.1 (iii) & M2 $\rightarrow$ $-ac \leq -bc$

3.2 (i) $\rightarrow$ $bc \leq ac$ $\blacksquare$

(iii) <u>~~If~~ $0 \leq a$ & $0 \leq b$ $\rightarrow$ $0 \leq ab$</u>

$0 \leq a$ & $0 \leq b$

O5 $\rightarrow$ $0 \cdot b \leq a \cdot b$

M2, 3.1 (ii) $\rightarrow$ $0 \leq ab$ $\blacksquare$

(iv) $0 \leq a^2$ $\forall a$

Proof by cases

<u>case 1:</u> $0 \leq a$

3.2 (iii) means $0 \leq a$ & $0 \leq a$ $\rightarrow$ $0 \leq a \cdot a = a^2$

<u>case 2:</u> $a \leq 0$

3.2 (i) $\rightarrow$ $0 \leq -a$

3.2 (iii) means $0 \leq -a$ & $0 \leq -a$ $\rightarrow$ $0 \leq (-a)(-a)$

3.1 (iv) $\rightarrow$ $0 \leq a^2$ $\blacksquare$

Thrm 3.2 (v) $\underline{0<1}$

$1 \in \mathbb{R}$ is distinct from 0

3.2 (iv) $\rightarrow$ $0 \leq 1^2 = 1 \cdot 1$

M3 $\rightarrow$ $0 \leq 1$

$0 \neq 1$ since $\mathbb{R}$ has distinct elements so

$0<1$ ☐

(vi) $\underline{0<a \rightarrow 0<a^{-1}}$

Consider $0<a$ &

Assume for sake of contradiction that $0 \geq a^{-1}$

$\underline{\text{Case 1: } a^{-1}=0}$

M2 $\rightarrow$ $a \cdot a^{-1} = a \cdot 0$

M4, 3.1(ii) $\rightarrow$ $1=0$ $\Rightarrow\Leftarrow$ since ~~[illegible]~~ $0<1$

$\underline{\text{Case 2: } a^{-1}<0}$: $\rightarrow a^{-1} \leq 0$

3.2(i) $0 \leq -a^{-1}$

Assume for sake of contradiction

$0 = -a^{-1}$

$-a^{-1} + a^{-1} = 0 + a^{-1}$

A2, A4, A3 $\rightarrow$ $0 = a^{-1}$ which is false since $a^{-1} < 0$ so

$0 < -a^{-1}$, using $0<a$

O5 $\rightarrow$ $0 \cdot a < -a^{-1} \cdot a$

M2, 3.1(ii), M4 $\rightarrow$ $0<-1 \rightarrow 0 \leq -1$

3.2(i) $\rightarrow$ $1 \leq 0$ $\Rightarrow\Leftarrow$ since $0<1$

$\therefore$ $0<a^{-1}$ must be true. ☐

(vii) ~~[illegible]~~ $\underline{0<a<b \rightarrow 0<b^{-1}<a^{-1}}$

$0<a<b \rightarrow 0<a$ & $0<b$

3.2(vi) $\rightarrow$ $0<a^{-1}$, $0<b^{-1}$

Separately $a<b$

$aa^{-1}b^{-1} < ba^{-1}b^{-1}$

M2, M4, M3 $\rightarrow$ $b^{-1}<a^{-1}$

Combining $b^{-1}<a^{-1}$ & $0<b^{-1}$ we get $0<b^{-1}<a^{-1}$ ☐