1. <u>Theorem</u>: Let $A, B \subset \mathbb{R}$ \& $A+B \equiv \{a+b \mid a \in A, b \in B\}$.
Prove that $\sup(A+B) = \sup A + \sup B$.

<u>Proof</u>: $C \equiv A+B$. Let a \& b be arbitrary elements s.t. $a \in A$, $b \in B$.

By the definition of supremum,

$$\sup A \geq a \;\; \& \;\; \sup B \geq b$$

Add to find $\sup A + \sup B \geq a+b$.
$\sup A + \sup B$ is an upper bound of C since $a+b$ is an arbitrary element of C. Therefore, $\underline{\sup A + \sup B \geq \sup C}$.
For any $\epsilon > 0$, $\exists$ some $a \in A$ s.t. $a > \sup A - \frac{\epsilon}{2}$.
If not then $\sup A - \frac{\epsilon}{2}$ is an upper bound of $A < \sup A$, which is a contradiction. Similarly, $\exists$ some $b \in B$ s.t. $b > \sup B - \frac{\epsilon}{2}$. Add these two to find

$$a+b > \sup A + \sup B - \epsilon.$$

This implies that $\underline{\sup C \geq \cancel{\sup A + \sup B}}$ since if we assume for the sake of contradiction that $\sup C < \sup A + \sup B$, then $\epsilon = \sup A + \sup B - \sup C$ would have no value $a+b = c$ s.t. $c > \sup A + \sup B - \epsilon$
$\rightarrow c > \sup C$ since $c \in C$ cannot be greater than $\sup C$.
Therefore $\sup C \geq \sup A + \sup B$. Combining this with ~~$\sup A + \sup B$~~ $\sup A + \sup B \geq \sup C$, we see that $\underline{\sup C = \sup A + \sup B}$.
$\blacksquare$

2. <u>Theorem</u>: Let $A, B \subseteq \mathbb{R}$, prove that $\sup(A \cup B) = \max(\sup A, \sup B)$.

<u>Proof</u>: $C \equiv A \cup B$.

Without loss of generality, assume $\max(\sup A, \sup B) = \sup B$. This implies $\sup B \geq \sup A$. Now consider an arbitrary element c s.t. $c \in C$. There are two cases.

<u>Case 1</u>: $c \in A$. By definition of supremum, $\sup A \geq c$. $\sup B \geq \sup A$ & $\sup A \geq c \to \sup B \geq c$ if $c \in A$.

<u>Case 2</u>: $c \in B$. By definition of supremum, $\sup B \geq c$ if $c \in B$.

Since $\sup B$ is greater than all elements in C, it is an upper bound. Therefore $\sup B \geq \sup C$.

Furthermore, <u>$\sup B \leq \sup C$</u> can be proven by contradiction.

<u>Assume $\sup C < \sup B$</u>. There must exist $b \in B$ s.t. $\sup C < b$. This is because if $\sup C \geq$ all $b \in B$ then $\sup C$ is an upper bound of B s.t. $\sup C < \sup B$. This is a contradiction so $\exists b$ s.t. $\sup C < b$. However, if $b \in B$, $b \in C$ by C's definition so $\sup C \geq b$. This is a contradiction with the premise so <u>$\sup B \leq \sup C$</u>. $\sup B \leq \sup C$ & $\sup C \leq \sup B \to \sup C = \sup B$.

$\therefore$ $\sup C = \max(\sup A, \sup B)$ since $\sup B$ was arbitrarily chosen as the maximum. ∎

3. 7.1 (a) $s_0, s_1, s_2, s_3, s_4 = 1, 1/4, 1/7, 1/10, 1/13$

(b) $-1, 4/3, 1, 10/11, 13/15$

(c) $0, 1/3, 2/9, 1/9, 4/81$

(d) $0, \sqrt{2}/2, 1, \sqrt{2}/2, 0$

7.2 (a) converges to 0

(b) converges to 3/4

(c) converges to 0

(d) does not converge, it oscillates.

7.3 (a) converges; 1

(b) converges; 1

(c) converges; 0

(d) converges; 1

(e) does not converge; oscillates

(f) converges; 1

(g) does not converge; diverges

(h) does not converge; oscillates diverges

(i) converges; 0

(j) converges; 7/2

(k) does not converge; diverges

(l) does not converge; oscillates

(m) converges; 0

(n) does not converge; oscillates

(o) does not converge; diverges

(p) converges; 2

(q) converges; 0

(r) converges; 1

(s) converges; 4/3

(t) converges; 0