9.2 $\lim x_n = 3$, $\lim y_n = 7$ & $y_n \neq 0 \ \forall n$

(a) $\lim(x_n + y_n) = \lim x_n + \lim y_n = 3 + 7 = \boxed{10}$

(b) $\lim\left(\frac{3y_n - x_n}{y_n^2}\right) = \frac{\lim(3y_n - x_n)}{\lim(y_n^2)} = \frac{\lim(3y_n) - \lim x_n}{(\lim y_n)(\lim y_n)}$

$= \frac{3\lim y_n - \lim x_n}{(\lim y_n)^2} = \frac{3 \cdot 7 - 3}{7^2} = \boxed{\frac{18}{49}}$

~~9.9 (c) Proving 8.9 (a) is a prerequisite to proving 9.9 (c),~~

~~8.9 (a) Theorem: If $s_n \geq a$ & (s_n) converges~~

8.9 (a) <u>Theorem</u>: Let (s_n) be a converging sequence. If $s_n \geq a$ for all but finitely many n, then $\lim s_n \geq a$.

<u>Proof</u>: Since $s_n \geq a$ for all but finitely many n, there exist finitely many n s.t. $s_n < a$. Let the set of all such n's be denoted

$$A = \{n_1, n_2, \ldots, n_k\} = \{n \mid s_n < a\} \text{ where } k \in \mathbb{N}.$$

Let $N_0 = \max A$. Since (s_n) converges, it is the case that for $\epsilon > 0$, $\exists N \in \mathbb{Z}^+$, s.t. for all $n > N$, $|s_n - s| < \epsilon$. Let $N \geq N_0$. Now assume for the sake of contradiction that $s < a$. This implies that $a - s > 0$. Let $a - s = \epsilon$ and substitute to find

$$|s_n - s| < a - s$$

$$\rightarrow \quad -a + s < s_n - s < a - s \quad \Big| \text{ add } s \text{ to all sides}$$

$$-a + 2s < s_n < a.$$

However, we have reached a contradiction with the statement $s_n < a$ since for all $n > N \geq N_0$, $s_n \geq a$ by N_0's definition. Therefore $s \geq a$. $\square$

9.9 (c) $\exists N_0$ s.t. $s_n \leq t_n$ for all $n > N_0$

Theorem: If $\lim s_n$ & $\lim t_n$ exist, then $\lim s_n \leq \lim t_n$

Proof: $t_n \geq s_n$ for all but finitely many n. ~~Assume~~ $t_n \geq s_n \to c_n \equiv t_n - s_n \geq 0$ for $n > N_0$. Assume (s_n) and (t_n) converge. c_n converges to $c = t - s$. From problem 8.9, we know that since $c_n \geq 0$ for $n > N_0$, $c \geq 0 \to t - s \geq 0 \to t \geq s$ or $\lim t_n \geq \lim s_n$ $\square$

9.15 Theorem: $\lim_{n \to \infty} \frac{a^n}{n!} = 0$ for $a \in \mathbb{R}$

Proof: ~~Since~~

Let $s_n = \frac{a^n}{n!}$.

$$L = \lim \left| \frac{s_{n+1}}{s_n} \right| = \lim \left| \frac{\frac{a^{n+1}}{(n+1)!}}{\frac{a^n}{n!}} \right| = \lim \left| \frac{a}{n+1} \right| = 0.$$

Since $L < 1$, $\lim s_n = 0$ from exercise 9.12. $\square$