10.1 (a) $a_n = \frac{1}{n}$.

for $n \neq 0$

$n+1 > n \to \frac{1}{n} > \frac{1}{n+1} \to a_n > a_{n+1}$ for $\forall n \in \mathbb{Z}_+$.

$\therefore$ (a_n) is decreasing

(a_n) is bounded since (a_n) is convergent. (specifically, it converges to 0). Let $\epsilon > 0$ and define $N \equiv \frac{1}{\epsilon}$. When $n > N$, $n > \frac{1}{\epsilon} \to \frac{1}{n} < \epsilon$ ~~for all n~~. Since $n > 0$, $\frac{1}{n} > 0$ so $|\frac{1}{n} - 0| < \epsilon$. $\therefore \lim a_n = 0$.

(b) $b_n = \frac{(-1)^n}{n^2}$.

Consider $b_n = \frac{(-1)^n}{n^2}$ & $b_{n+1} = \frac{(-1)^{n+1}}{(n+1)^2}$. If n is even, $b_n > b_{n+1}$. If n is odd, b_{n+1} ~~...~~ $> b_n$. Therefore, (b_n) is neither increasing or decreasing. b_n is bounded. Consider two sequences (x_n) and (y_n) where $x_n = \frac{1}{n^2}$ & $y_n = \frac{-1}{n^2}$. Furthermore $X = \{x_n\}$, $Y = \{y_n\}$ and $B = \{b_n\}$. x_n & y_n can be shown to converge to 0 for $N \equiv \frac{1}{\sqrt{\epsilon}}$ so x_n & y_n are bounded and X & Y have supremums & infimums. ~~Therefore~~ Now consider ~~...~~ $X \cup Y$. $\max\{\sup X, \sup Y\}$ is an upper bound and $\min\{\inf X, \inf Y\}$ is a lower bound of $X \cup Y$. Therefore $\max\{\sup X, \sup Y\}$ is an upper bound of B and $\min\{\inf X, \inf Y\}$ is a lower bound for B since $B \subseteq X \cup Y$. $\therefore$ b_n is bounded.

(c) $n^5 = c_n$

(c_n) is increasing since $n < n+1 \to n^5 < (n+1)^5 \to c_n < c_{n+1}$.

(c_n) is unbounded. ~~because~~ ~~...~~ Assume for sake of contradiction that there existed an upper bound, u, ~~...~~ $\exists v \in \mathbb{Z}^+$ s.t. $v > u$. $v^5 = c_v$ and $v^5 > u$ so no such upper bound exists.

10.1 (d) $d_n = \sin\left(\frac{n\pi}{7}\right)$

(d_n) is neither increasing or decreasing since $d_1 < d_3$ but $d_3 > d_5$. (d_n) is bounded because the sin function is bounded such that $-1 \leq \sin x \leq 1$.

(e) $e_n = (-2)^n$

(e_n) is neither increasing or decreasing since ~~$d_1 < d_2$ but $d_2 > d_3$~~ $e_1 < e_2$ but $e_2 > e_3$. (e_n) is unbounded since if there existed some upper bound u, we could select some even v s.t. $v > u$ and $(-2)^v = 2^v > u$.

(f) $\frac{n}{3^n} = f_n$

(f_n) is decreasing since if we divide $\frac{f_{n+1}}{f_n} = \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3}} = \frac{n+1}{3n}$.

$n \geq 1 \rightarrow 2n \geq n+1 \rightarrow 2 \geq \frac{n+1}{n} \rightarrow \frac{2}{3} \geq \frac{n+1}{3n}$ which implies $\frac{f_{n+1}}{f_n} < 1 \rightarrow f_{n+1} < f_n$. (f_n) is bounded. (f_n) converges to zero by performing the ratio test. $L = \lim_{n\to\infty}\left|\frac{f_{n+1}}{f_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{3n}\right|$

$L = \frac{1}{3} < 1$, $\therefore$ (f_n) converges $\rightarrow$ (f_n) is bounded.

10.6 (a) (s_n) is a sequence s.t. $|s_{n+1} - s_n| < 2^{-n}$ for all $n \in \mathbb{N}$

Theorem: (s_n) is a cauchy sequence

Proof: Consider the sequence (ϵ_n) s.t. $\epsilon_n = \frac{1}{2^n} = 2^{-n}$.

From theorem 9.7 we know (ϵ_n) is a convergent sequence since $\frac{1}{2^n} = \left(\frac{1}{2}\right)^n$ and $\left|\frac{1}{2}\right| < 1$. Also, (ϵ_n) converges to 0. Therefore, let $\epsilon > 0$. For some $n > N$

$$\left|\frac{1}{2^n} - 0\right| < \epsilon \to \frac{1}{2^n} < \epsilon$$

$$\to |s_{n+1} - s_n| < \epsilon.$$

Let $m, n > N$ and WLOG, we can assume $m \geq n$. I will now prove the statement

$P(k)$ = "$|s_m - s_n| < \epsilon$ for $m = n+k$", for all $k \in \mathbb{N}$

Base case: $P(1)$, $P(0)$.

$P(0)$ is the trivial statement that $0 < \epsilon$.

$P(1)$ follows from the work done above.

Induction Step:

Assume $P(k)$ so $|s_{n+k} - s_n| < \epsilon$. From $P(1)$, we know $|s_{n+k+1} - s_{n+k}| < \epsilon$. Add these to get

$$|s_{n+k} - s_n| + |s_{n+k+1} - s_{n+k}| < 2\epsilon$$

And use the triangle inequality

$$|s_{n+k+1} - s_{n+k} + s_{n+k} - s_n| \leq |s_{n+k} - s_n| + |s_{n+k+1}| < 2\epsilon$$

$$\to |s_{n+k+1} - s_n| < 2\epsilon.$$

This is the statement $P(k) \to P(k+1)$ so by the principle of mathematical induction, $P(k)$ for all $k \in \mathbb{N}$.

Therefore $|s_m - s_n| < \epsilon$ when $\epsilon > 0$ for some $n > N$.

$\to$ (s_n) is a cauchy sequence. $\square$

(b) From theorem 9.7, we know (r_n), where $r_n = \frac{1}{n}$, converges to 0. The rest of the proof is unchanged.

10.7 S is a bounded nonempty subset of $\mathbb{R}$ such that $\sup S$ is not in S. Prove there is a sequence (s_n) of points in S such that $\lim s_n = \sup S$.

Pf: Let S be a bounded nonempty subset of $\mathbb{R}$ such that $\sup S \notin S$. Let (t_n) where $t_n = \sup S$ and define (u_n) s.t. $u_n = t_n - \frac{1}{n}$. From the definition of supremum, $\sup S \geq s$ for all $s \in S$. Since $s \neq \sup S$ we can say $\sup S > s \ \forall s \in S$. By def'n of $\sup S$, $\exists s \in S$ s.t. $s > \sup S - \epsilon$ ~~Since~~ $\cancel{\frac{1}{n}}$ where $\epsilon > 0$. Since $1/n > 0$, let $\epsilon = 1/n$. And define s_n s.t. $s_n \in S$ and $s_n > \sup S - \frac{1}{n} \to s_n > u_n$. Observe $u_n < s_n < t_n$. Since $(u_n), (t_n)$ converge to $\sup S$, (s_n) converges to $\sup S$ by the squeeze lemma.

Proof:

10.8 Claim: (s_n) is an increasing sequence of positive numbers $\ni$ $\sigma_n = \frac{1}{n}(s_1 + s_2 + \dots + s_n) \rightarrow s_n \geq \sigma_n$

Pf: Base Step: $n=1$

$$\sigma_1 = s_1 \leq s_1$$

Induction Step $\sigma_n \leq s_n \rightarrow \sigma_{n+1} \leq s_{n+1}$

Assume $\sigma_n \leq s_n$

$$\frac{1}{n}(s_1 + \dots + s_n) \leq s_n \rightarrow \frac{1}{n}(s_1 + \dots + s_n) \leq s_{n+1}$$

$$s_1 + \dots + s_n + s_{n+1} \leq n s_{n+1} + s_{n+1} = (n+1) s_{n+1}$$

$$\frac{1}{n+1}(s_1 + \dots + s_{n+1}) \leq s_{n+1} \rightarrow \sigma_{n+1} \leq s_{n+1}.$$

By the induction hypothesis, $\sigma_n \leq s_n \ \forall n$.

Rest of Proof: WTS $\sigma_{n+1} \geq \sigma_n$

Base Step: $n=1$ WTS $\sigma_2 \geq \sigma_1$

$$\sigma_1 = s_1 \quad , \sigma_2 = (s_1 + s_2) \cdot \frac{1}{2}$$

$$2(\sigma_2 - \sigma_1) = s_2 - s_1$$

$$s_2 \geq s_1 \rightarrow s_2 - s_1 \geq 0$$

$$\text{so } 2(\sigma_2 - \sigma_1) \geq 0 \rightarrow \sigma_2 \geq \sigma_1$$

Induction Step: Assume $\sigma_{n+1} \geq \sigma_n$

Now consider $\sigma_{n+2} - \sigma_{n+1}$:

$$\sigma_{n+2} - \sigma_{n+1} = \frac{1}{n+2}(s_1 \dots s_{n+1} + s_{n+2}) - \frac{1}{n+1}(s_1 + \dots + s_{n+1})$$

$$\sigma_{n+2} - \sigma_{n+1} = \frac{1}{n+2} s_{n+2} - \frac{1}{(n+2)(n+1)}(s_1 \dots s_{n+1})$$

$$(n+2)(\sigma_{n+2} - \sigma_{n+1}) = s_{n+2} - \sigma_{n+1}$$

From above $\sigma_{n+1} \leq s_{n+1} \leq s_{n+2}$ so

$$(n+2)(\sigma_{n+2} - \sigma_{n+1}) \geq 0$$

$$\rightarrow \sigma_{n+2} \geq \sigma_{n+1}.$$

By the induction method $\sigma_{n+1} \geq \sigma_n \ \forall n$. Therefore (σ_n) is an increasing sequence.