MATH 104 Homework 1

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1 Ross 1.10

We can rewrite $(2n+1)+(2n+3)+...+(4n-1)$ as $\sum_{i=1}^{n} 2n+(2i-1)$ and proceed to prove by induction.

The base case is $n=1$. $\sum_{i=1}^{n} 2n+(2i-1)|_{n=1}=3$, which is equivalent to $3n^2|_{n=1}=3$ which is true.

For the induction hypothesis, we assume that this holds for $n=k$, and seek to show that it also holds for $n=k+1$. Assuming $\sum_{i=1}^{n} 2n+(2i-1)|_{n=k}=3k^2$, we prove $\sum_{i=1}^{n} 2n+(2i-1)|_{n=k+1}=3(k+1)^2$ through the following

$$
\begin{aligned}
\sum_{i=1}^{n} 2n+(2i-1)|_{n=k+1} &= \sum_{i=1}^{k+1} 2(k+1)+(2i-1) \\
&= \left\{\sum_{i=1}^{k} 2(k+1)+(2i-1)\right\}+4k+3 \\
&= \left\{\sum_{i=1}^{k} 2k+(2i-1)\right\}+\sum_{i=1}^{k} 2+4k+3 \\
&= \left\{\sum_{i=1}^{k} 2k+(2i-1)\right\}+6k+3 \\
&= 3k^2+6k+3 \\
&= 3(k+1)^2
\end{aligned}
$$

2 Ross 1.12

1. (a) $n=1$

$$
\begin{aligned}
(a+b)^1 &= \binom{1}{0}a^1+\binom{1}{1}a^{1-1}b \\
&= a^1+1a^{1-1}b \\
&= a+b
\end{aligned}
$$

(b) $n = 2$

$$\begin{aligned}(a+b)^2 &= \binom{2}{0}a^2 + \binom{1}{1}a^{2-1}b + \binom{2}{2}a^{2-2}b^2 \\ &= a^2 + 2a^{2-1}b + \frac{1}{2}(2)(1)a^{2-2}b^2 \\ &= a^2 + 2ab + b^2\end{aligned}$$

(c) $n = 3$

$$\begin{aligned}(a+b)^3 &= \binom{3}{0}a^3 + \binom{3}{1}a^{3-1}b + \binom{3}{2}a^{3-2}b^2 + \binom{3}{3}a^{3-3}b^3 \\ &= a^3 + 3a^{3-1}b + \frac{1}{2}(3)(3-1)a^{3-2}b^2 + b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

2. To show $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ for $k = 1, 2, ..., n$ we have the following:

$$\begin{aligned}\binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\ &= \frac{n-k+1}{n-k+1}\frac{n!}{k!(n-k)!} + \frac{k}{k}\frac{n!}{(k-1)!(n-k+1)!} \\ &= \frac{(n+1-k)n! + kn!}{k!(n+1-k)!} \\ &= \frac{(n+1)n!}{k!((n+1)-k)!} \\ &= \frac{(n+1)!}{k!((n+1)-k)!} \\ &= \binom{n+1}{k}\end{aligned}$$

3. We've already verified the binomial theorem for the base case $n = 1$. Let us assume that for the n case the binomial theorem is true, and then prove that it holds for the $n + 1$ case.

For the n case, assume

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + ... + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n.$$

Let us show that that for the $n + 1 case$,

$$(a+b)^(n+1) = \binom{n+1}{0}a^{n+1} + \binom{n+1}{1}a^nb + ... + \binom{n+1}{n}ab^{n-1} + \binom{n+1}{n+1}b^{n+1}.$$

We proceed as follows:

$$
\begin{aligned}
(a+b)^{(n+1)} &= (a+b)(a+b)^n \\
&= (a+b)\left\{\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + ... + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n\right\} \\
&= a\left\{\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + ... + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n\right\} + \\
&\qquad b\left\{\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + ... + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n\right\} \\
&= \left\{\binom{n}{0}a^{n+1} + \binom{n}{1}a^{n}b + ... + \binom{n}{n-1}a^2b^{n-1} + \binom{n}{n}ab^n\right\} + \\
&\qquad \left\{\binom{n}{0}a^{n}b + \binom{n}{1}a^{n-1}b^2 + ... + \binom{n}{n-1}ab^{n} + \binom{n}{n}b^{n+1}\right\} \\
&= \binom{n}{0}a^{n+1} + \sum_{i=1}^{n}\left\{\binom{n}{i} + \binom{n}{i-1}\right\}a^{n+1-i}b^i + \binom{n}{n}b^{n+1} \\
&= \binom{n}{0}a^{n+1} + \sum_{i=1}^{n}\left\{\binom{n+1}{i}\right\}a^{n+1-i}b^i + \binom{n}{n}b^{n+1} \\
&= \binom{n}{0}a^{n+1} + \binom{n+1}{1}a^{n}b + ... + \binom{n+1}{n}ab^{n-1} + \binom{n}{n}b^{n+1} \\
&= \binom{n+1}{0}a^{n+1} + \binom{n+1}{1}a^{n}b + ... + \binom{n+1}{n}ab^{n-1} + \binom{n+1}{n+1}b^{n+1}
\end{aligned}
$$

Hence we have proven the binomial theorem using induction and part (b) on the third to last step.

3 Ross 2.1

Using the rational zeros theorem:

1. $x^2 - 3 = 0$: The only possible rational roots are $\pm 1, \pm 3$, neither of which solve the equation.
2. $x^2 - 5 = 0$: The only possible rational roots are $\pm 1, \pm 5$, neither of which solve the equation.
3. $x^2 - 7 = 0$: The only possible rational roots are $\pm 1, \pm 7$, neither of which solve the equation.
4. $x^2 - 24 = 0$: The only possible rational roots are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$, none of which solve the equation.
5. $x^2 - 31 = 0$: The only possible rational roots are $\pm 1, \pm 31$, neither of which solve the equation.

4 Ross 2.2

Using the rational zeros theorem:

1. $x^3 - 2 = 0$: The only possible rational roots are $\pm 1, \pm 2$, neither of which solve the equation.
2. $x^7 - 5 = 0$: The only possible rational roots are $\pm 1, \pm 5$, neither of which solve the equation.
3. $x^4 - 13 = 0$: The only possible rational roots are $\pm 1, \pm 13$, neither of which solve the equation.

5 Ross 2.7

1.

$$\begin{aligned}\sqrt{4+2\sqrt{3}} - \sqrt{3} &= x \\ \sqrt{4+2\sqrt{3}} &= x + \sqrt{3} \\ (\sqrt{4+2\sqrt{3}})^2 &= (x+\sqrt{3})^2 \\ 4 + 2\sqrt{3} &= x^2 + 2\sqrt{3}x + 3\end{aligned}$$

It can be readily seen that 1 solves this expression, hence $\sqrt{4+2\sqrt{3}} - \sqrt{3} = 1$ and is rational.

2.

$$\begin{aligned}\sqrt{6+4\sqrt{2}} - \sqrt{2} &= x \\ \sqrt{6+4\sqrt{2}} &= x + \sqrt{2} \\ (\sqrt{6+4\sqrt{2}})^2 &= (x+\sqrt{2})^2 \\ 6 + 4\sqrt{2} &= x^2 + 2\sqrt{2}x + 2\end{aligned}$$

It can be readily seen that 2 solves this expression, hence $\sqrt{6+4\sqrt{2}} - \sqrt{2} = 2$ and is rational.

6 Ross 3.6

The triangle inequality states that $|a+b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$. As the hint says, we will apply this twice to prove that $|a+b+c| \leq |a| + |b| + |c|$.

1. Let us group together two of the terms and apply triangle inequality. We know that $|(a+b)+c| \leq |a+b|+|c|$. In addition, we know that $|a+b| \leq |a|+|b|$. Combining these two terms together, we have $|(a+b)+c| \leq |a+b|+|c| \leq |a|+|b|+|c|$ and have hence proven the statement.

2. To prove the general case, with induction, we already have proven the base case with the triangle inequality. Hence moving onto the induction hypothesis, we wish to prove that if this hold for the n terms, then it will also hold for $n+1$ terms. In the n-th case, we assume that:

$$|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$$

We wish to prove the following with this assumption:

$$|a_1 + a_2 + ... + a_n + a_{n+1}| \leq |a_1| + |a_2| + ... + |a_{n+1}|.$$

We can do so by once again grouping them into two separate terms. Let the first term be the first n elements, and the second term be a_{n+1}. What we have is something similar to what we proved in part 1. We know the following through the triangle inequality:

$$|(a_1 + a_2 + ... + a_n) + a_{n+1}| \leq |(a_1 + a_2 + ... + a_n)| + |a_{n+1}|$$

And from our induction hypothesis we know that:

$$|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|.$$

Combining these two inequalities we have

$$|(a_1+a_2+...+a_n)+a_{n+1}| \leq |(a_1+a_2+...+a_n)|+|a_{n+1}| \leq |a_1|+|a_2|+...+|a_n|+|a_{n+1}|$$

which concludes the proof.

7 Ross 4.11

The denseness property of $\mathbb{Q}$ states that if $a, b \in \mathbb{R}$ there is a rational $r \in \mathbb{Q}$ such that $a < r < b$. We can prove that there are an infinite amount of rationals between a and b by repeatedly using the denseness property between either a and b with the rational value r that we discover. For any arbitrary n, we can find n number of rationals between a and b. The base case where $n = 1$ is given by the denseness property, since we know there exists at least 1 r between a and b. Now on the inductive hypothesis, assume have already found n such rational numbers between a and b, discovered in such a way each r_i is between a and r_{i-1}, that is we keep finding rationals that get progressively closer to a (this choice is arbitrary, any pair that has not already been searched between will work). Assuming we have already found r_n, which is the n-th rational number between a and b, we can find the $n+1$-th rational number by applying the denseness property onto a and r_n. There must exist a rational r_{n+1} number such that $a < r_{n+1} < r_n$, hence we have proven the inductive step and shown that there are an infinite number of rational numbers, as we can arbitrarily increase n.

8 Ross 4.14

1. First, we show that for any $b \in B$, $\sup(A+B) - b$ is an upper bound for the set A. In order to show this, we need to show that $\sup(A+B) - b \geq a$ for all $a \in A$. By rearranging this inequality, we have $\sup(A+B) \geq a + b$ for a given $b \in B$ and any $a \in A$. This is by definition true, since $A + B$ is defined to be the set of all sums $a + b$ where $a \in A$ and $b \in B$. Since $\sup(A+B) - b$ is an upper bound for A, as the hint suggests, this implies that $\sup(A+B) - b$ is greater than $\sup A$. Since we know the inequality $\sup(A+B) - b \geq \sup A$ for any $b \in B$, we can rearrange this inequality to read $\sup(A+B) - \sup A \geq b$ for any $b \in B$. By definition, this means that $\sup(A+B) - \sup A$ is an upper bound for B, and hence $\sup(A+B) - \sup A \geq \sup B$. As a result, $\sup A + \sup B \leq \sup(A+B)$.

   We know separately that $\sup A \geq a$ for all $a \in A$, and $\sup B \geq b$ for all $b \in B$. Combining these together, we know that $\sup A + \sup B$ is an upperbound for the sum of any $a \in A, b \in B$. Since $\sup A + B$ is defined to be the least upper bound, it must be the case that $\sup A + B \leq \sup A + \sup B$. However, since we showed before that $\sup A + \sup B \leq \sup(A+B)$, these two statements combined indicate that it must be that case that $\sup A + \sup B = \sup{(A+B)}$

2. For any given $b \in B$, $\inf(A+B) - b \leq a$ for all $a \in A$ which can be shown by simply moving b to the other side and observing this is the definition of the infimum of the set $A + B$. This indicates that $\inf(A+B) - b$ is a lower bound for a and hence, $\inf(A+B) - b \leq \inf a$. By moving things around, we also have that for all $b \in B$, $\inf(A+B) - \inf a \leq b$, and hence by similar reasoning, $\inf(A+B) - \inf a \leq \inf b$. As a result, we have $\inf(A+B) \leq \inf b + \inf a$.

   We know that separately $\inf A \leq a$ for all $a \in A$, and $\inf B \leq b$ for all $b \in B$. Combining these together, we know that $\inf A + \inf B$ is an lowerbound for any sum of $a \in A, b \in B$. Since $\inf A + B$ is defined to be the greatest lower bound, it must be the case that $\inf A + \inf B \leq \inf A + B$. However, since we showed before that $\inf(A+B) \leq \inf b + \inf a$, these two statements combined indicate that it must be that case that $\inf A + \inf B = \inf{(A+B)}$

9 Ross 7.5

1. $\lim s_n$ where $s_n = \sqrt{n^2+1} - n$

$$\begin{aligned}
\sqrt{n^2+1} - n &= \left(\sqrt{n^2+1} - n\right) \frac{\left(\sqrt{n^2+1}+n\right)}{\left(\sqrt{n^2+1}+n\right)} \\
&= \frac{n^2+1-n}{\sqrt{n^2+1}+n} \\
&= \frac{1}{\sqrt{n^2+1}+n} \\
&= 0 \text{ as } n \text{ approaches } \infty
\end{aligned}$$

2. $\lim \left(\sqrt{n^2+n} - n\right)$

$$\begin{aligned}
\left(\sqrt{n^2+n} - n\right) &= \left(\sqrt{n^2+n} - n\right) \frac{\left(\sqrt{n^2+n}+n\right)}{\left(\sqrt{n^2+n}+n\right)} \\
&= \frac{n^2+n-n^2}{\sqrt{n^2+1}+n} \\
&= \frac{n}{\sqrt{n^2+1}+n} \\
&= \frac{1}{2} \text{ as } n \text{ approaches } \infty \text{ since } \sqrt{n^2+1} \text{ approaches } n
\end{aligned}$$

3. $\lim \left(\sqrt{4n^2+n} - 2n\right)$

$$\begin{aligned}
\left(\sqrt{4n^2+n} - 2n\right) &= \left(\sqrt{4n^2+n} - 2n\right) \frac{\left(\sqrt{4n^2+n}+2n\right)}{\left(\sqrt{4n^2+n}+2n\right)} \\
&= \frac{4n^2+n-4n^2}{\left(\sqrt{4n^2+n}+2n\right)} \\
&= \frac{n}{\left(\sqrt{4n^2+n}+2n\right)} \\
&= \frac{n}{2n\sqrt{1+\frac{1}{4n}}+2n} \\
&= \frac{n}{4n} \\
&= \frac{1}{4} \text{ as } n \text{ approaches } \infty \text{ since } \sqrt{1+\frac{1}{4n}} \text{ approaches } 1
\end{aligned}$$