MATH104 Homework 2

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1 Ross 9.9

1. The definition of lim $s_n = +\infty$ is that for each $M > 0$, there is a number N such that $n > N$ implies $s_n > M$. If the limit of s_n is $+\infty$, starting from N_0 we can apply this definition through s_n onto t_n. We know that for all $n > N_0$, $t_n \geq s_n$, and for some M, there exists N such that $s_n > M$. If we consider all $n > N_0 \cap n > N$, then we know that $t_n \geq s_n$ as $n > N_0$, and $s_n > M$ as $n > N$, which implies $t_n \geq s_n > M$ therefore proving that lim $t_n = +\infty$

2. Using similar logic as the previous part, if we know that if lim $t_n = -\infty$, then there exists a number N for each $M < 0$ such that all $n > N$ implies $t_n < M$. In addition, there exists some N_0 such that $s_n \leq t_n$ for all $n > N_0$. If we consider all n such that $n > N_0$ and $n > N$, then we know that $s_n \leq t_n$, and $s_n < M$, and hence $s_n \leq t_n \leq M$ for all n that satisfy the condition. Since for all aforementioned n, s_n is less than M for any $M < 0$, s_n diverges to $-\infty$

3. If lim s_n and lim t_n exist, then we can prove that lim t_n - lim $s_n \geq 0$. Let the limits be represented at a and b respectively. In lecture we proved lim $(s_n + t_n) = a + b$, and similarly, we know that lim $(t_n - s_n) = b - a$. Using this definition, we can use the fact that there exists N_0, such that for all $n > N_0$, we will have $s_n \leq t_n$ or that $t_n - s_n \geq 0$. Take the maximum of N and N_0, where N represents the value in the proof in lecture where $n > N$ implies $|(t_n - s_n) - (b - a)| \leq \epsilon$. If we consider all n such that $n > N_0$ and $n > N$, we know that the $t_n - s_n \geq 0$, and the difference between these two sequences and their limits will be less than any $\epsilon > 0$, hence it must be the case that the lim t_n - lim $s_n \geq 0$

2 Ross 9.15

$$\begin{aligned}
\lim_{n\to\infty}\frac{a^n}{n!} &= \frac{a \times a \times a \times ...}{n \times (n-1) \times ... \times a \times ... \times 1} \\
&= \frac{a^a}{a!} \times \frac{a^{n-a}}{\frac{n!}{(a)!}} \\
&= \frac{a^k}{k!} \times \frac{a}{a+1} \times \frac{a}{a+2} \times ...
\end{aligned}$$

We can treat the first term as a constant. For all $a \in R$, there is some $k \in \mathbb{N}$ such that $a > k > a+1$, where the first k terms in this product will be products of terms greater than 1, represented as $\frac{a^k}{k!}$, and the last $n-k$ terms will be products of terms less than 1. Consider the sequence a_n which represents these terms. For any $\epsilon > 0$, we can find such N such that for all $n > N$, $|a_n| <= \epsilon$. Since we are multiplying a constant term $\frac{a^k}{k!}$ by many fractions smaller than 1, we can simply keep increasing n which means multiplying by smaller and smaller fractions get below ϵ. By definition, this means the sequence converges to 0.

3 Ross 10.7

Consider the sequence $s'_n = \sup S - \frac{1}{n}$. Since we can subtract an arbitrarily small amount from $\sup S$, there will exist a value s_n for every index n such that $s'_n \leq s_n \leq \sup S$. Hence, if we consider all points in S such that it is greater than s'_n, then it will converge to $\sup S$, as we know there exists M such that any $n > M$ we will have $\sup S - \frac{1}{n} \leq s_n < \sup S$, which implies that $0 > \sup S - s_n \geq \frac{1}{n}S$. Since this works for any large n, we have that the limit of s_n is $\sup S$ since we can find that the distance is less than any $\epsilon > 0$ by just increasing n.

4 Ross 10.8

We will prove that $\sigma_{n+1} \geq \sigma_n$

$$\begin{aligned}
\sigma_n &\leq \sigma_{n+1} \\
\frac{1}{n}(s_1 + s_2 + ... + s_n) &\leq \frac{1}{n+1}(s_1 + s_2 + ... + s_{n+1}) \\
(n+1)(s_1 + s_2 + ... + s_n) &\leq n(s_1 + s_2 + ... + s_{n+1}) \\
(s_1 + s_2 + ... + s_n) &\leq n(s_{n+1}) \\
(s_1 + s_2 + ... + s_n) &\leq s_{n+1} + s_{n+1} + ... + s_{n+1}
\end{aligned}$$

From the last line, an element wise comparison between s_i and s_{n+1} for $i < n$ verifies the inequality, since s_n is an increasing sequence and that implies that $s_n \leq s_{n+1}$ for all n.

5 Ross 10.9

$$s_1 = 1, s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2 \forall n \geq 1$$

1. $s_2 = \frac{1}{2}(1)^2 = \frac{1}{2}$,
   $s_3 = \frac{2}{3}(\frac{1}{2}^2) = \frac{1}{6}$
   $s_4 = \frac{3}{4}(\frac{1}{6}^2) = \frac{1}{48}$
2. The sequence is bounded from above by 1 and bounded from below by zero since both terms will always be positive. In addition, $\frac{n}{n+1}$ and s_n^2 are decreasing. A decreasing bounded sequence must converge.
3. Consider $s'_n = \frac{1}{n}$. By inspection, $s'_n \geq s_n \geq 0$ for all $n \geq 1$. We know s'_n converges to 0, and since s_n is between s'_n and 0, which lower bounds both of them, it must converge to the limit of 0 and the limit of s'_n, which is 0.

6 Ross 10.10

$$s_1 = 1, s_{n+1} = \frac{1}{3}(s_n + 1) \forall n \geq 1$$

1. $s_2 = \frac{2}{3}$,
   $s_3 = \frac{1}{3}(\frac{2}{3} + 1) = \frac{5}{9}$
   $s_4 = \frac{1}{3}(\frac{5}{9} + 1) = \frac{14}{27}$
2. In the base case, $s_1 = 1$ and hence satisfies $s_n > \frac{1}{2}$. Assume $s_n > \frac{1}{2}$, show that $s_{n+1} > \frac{1}{2}$ as well.

$$s_{n+1} = \frac{1}{3}(s_n + 1) < \frac{1}{3}(\frac{1}{2} + 1) = \frac{1}{2}$$

3. We want to show $s_{n+1} \leq s_n$

$$
\begin{aligned}
s_n &\geq s_{n+1} \\
s_n &\geq \frac{1}{3}(s_n + 1) \\
3s_n &\geq (s_n + 1) \\
2s_n &\geq 1 \\
s_n &\geq \frac{1}{2}
\end{aligned}
$$

Since all things are iff, the first line is true and hence s_n is a decreasing statement.

4. s_n is decreasing and bounded therefore must have converge to a value.

$$s = \frac{1}{3}(s+1) \implies s = \frac{1}{2}$$

7 Ross 10.11

1. t_n is upper bounded by 1 and lower bounded by zero. $\left[1 - \frac{1}{4n^2}\right]$ is always less than 1 for all $n > 0$, and as a result, t_{n+1} will always be the previous term multiplied by a smaller positive term. Hence it is also decreasing. As a result, it must have a limit since it is bounded and decreasing.

2. Not zero since the discounting over each term gets smaller. Probably something weird and irrational.

8 Squeeze Test

Let a_n, b_n, c_n be three sequences where $a_n \leq b_n \leq c_n$ and $L = \lim\, a_n = \lim\, c_n$. Then $\lim b_n = L$ as well because for a_n and c_n, we know that there exists some N for any $\epsilon > 0$ where for all $n > N$, both $|a_n - L| < \epsilon$ and $|c_n - L| < \epsilon$ by definition of limits (N is max of (N_a, N_c)). Expanding the absolute value we have:

$$
\begin{gathered}
-\epsilon < a_n - L \leq c_n - L \leq \epsilon \\
L - \epsilon < a_n \leq c_n \leq L + \epsilon \\
\implies L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon \\
\implies L - \epsilon < b_n < L + \epsilon \\
\implies |b_n - L| < \epsilon
\end{gathered}
$$

Which concludes the proof.