Math 104 Homework 4

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1 Ross 12.10

From the reverse direction, if $\limsup |s_n| < +\infty$, then we know that for any $N \in \mathbb{N}$, $\limsup s_n > s_n$ for $n > N$ and hence upper bounds the tail and is not ∞. In addition, we can take $M_1 = \max(s_1, ..., s_{N-1})$ to be an upper bound for the first chunk. Therefore, $|s_n|$ is bounded by $\max(\limsup s_n, M_1)$. From the forward direction, if s_n is bounded, the values of s_n will never go outside the bounds, therefore the $\limsup \neq \infty$. Hence the statement is iff.

2 Ross 12.12

1. The middle inequality is true by definition, hence we only prove the first and last inequality. For the first inequality, we propose an intuitive argument. First, let us argue that $\inf\{\sigma_n : n > N\} \geq \inf\{s_n : n > n\}$. Since σ is the average of the terms up until n, it will always be larger than or equal to the small term up until n. We can prove this by contradiction, suppose the average μ is smaller than the smallest value $\underline{x}$, then that means there must be smaller values pulling the average down, hence $\underline{x}$ is not the smallest value. Since this holds true for all subsets of the sequence s_n, then it also holds as $N \to \infty$. Through a similar argument, we can prove that the $\sup \sigma_n$ is less than $\sup s_n$. Averaging the values of s_n will cause the largest value of σ_n to be dragged down by small values, and hence will necessarily be smaller than $\sup s_n$.

2. If $\lim s_n$ exists, then s_n converges. Therefore, we can treat σ_n as a series. Since s_n converges to s, we can show that $\sigma_n = \frac{1}{n} \sum s_n$, and as this is an average of the terms of s_n, as s_n converges, then σ_n will also converge to s, as taking the average to infinity we have an infinite sum of values close to s divided by an infinite amount. $\frac{s\infty}{\infty} = s$, hence $\lim \sigma_n = \lim s_n = s$

3. Any sequence which alternates between constant terms will have σ_n converge to zero (as consecutive terms cancel each other out, and the limit will be zero).

3 Ross 14.2

1. $$\sum \frac{n-1}{n^2}$$

   Diverges because in limit this simplifies to $\frac{1}{n}$, which we know diverges.

2. $$\sum (-1)^n$$

   Diverges since this alternates between 1 and 0.

3. $$\sum \frac{3n}{n^3}$$

   Converges because this simplifies to $\frac{3}{n^2}$, and we know $\frac{1}{n^2}$ converges

4. $$\sum \frac{n^3}{3^n}$$

   $|a_n|^{\frac{1}{n}} = |\frac{n^{3/n}}{3^{n/n}}| = n^{\frac{3}{n}}$

   Converges because this is the product of $n^{\frac{1}{n}}$ which we know converges.

5. $$\sum \frac{n^2}{n!}$$

   $|\frac{a_{n+1}}{a_n}| = \left|\frac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}}\right| = \frac{(n+1)^2}{n^2(n+1)} \approx \frac{1}{n+1} \approx 0$

   Converges by ratio test.

6. $$\sum \frac{1}{n^n}$$

   $|a_n|^{1/n} = |\frac{1}{n}|$

   Converges by root test.

7. $$\sum \frac{n}{2^n}$$

   $|a_n|^{1/n} = |\frac{n^{\frac{1}{n}}}{2}| \approx \frac{1}{2}$

   Converges by root test.

4 Ross 14.10

Since by Example 8, we know that $a_n = 2^{(-1)^n - n}$ converges, let us replace 2 with 1/2 and hope that it works. We have a similar result as Example 8, for even n, $|a_{n+1}/a_n| = 2$ and for odd terms it equal 1/8, hence the ratio gives no information similar to example 8. For the root test however, we see that $\alpha = |a_n|^{1/n} = \left(\frac{1}{2}^{(-1)^n - n}\right)^{1/n}$ is equal to $\frac{1}{2}^{1/n-1}$ for even terms and $\frac{1}{2}^{-(1/n)-1}$ for odd terms, both of which converge to 2. Therefore by the root test, this sequence diverges.

5 Rudin 3.6

1. $$a_n = \sqrt{n+1} - \sqrt{n}$$

   Notice that $a_n + a_{n+1} = \sqrt{n+1} - \sqrt{n} + \sqrt{n+2} - \sqrt{n+1} = \sqrt{n+2} - \sqrt{n}$. By repeatedly expanding the sum until $n = 1$, we have $\sum_N a_n = \sqrt{N+1} - 1$, which approaches infinity as $N \to \infty$

2. $$a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}$$

   $$a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{n\sqrt{n+1} + n^{3/2}}$$

   $a_n < s_n = \frac{1}{n^{3/2}}$ since $n\sqrt{n+1} > 0$ for all n. Hence by comparison test, since $\frac{1}{n^{3/2}}$ converges we know that a_n converges as well.

3. $$a_n = (n^{1/n} - 1)^n$$

   By root test, $\alpha = |(n^{1/n} - 1)^n|^{1/n} = |(n^{1/n} - 1)|$. We know that $(n^{1/n}$ converges to 1, hence $\limsup \alpha = 0$ and therefore this converges.

6 Rudin 3.7

If a_n converges, then we know that by the ratio test $\limsup |a_n|^{1/n} < 1$ Therefore, for $b_n = \frac{\sqrt{a_n}}{n}$, if we apply the ratio test we have $|\frac{\sqrt{a_n}}{n}|^{1/n} = |\frac{a_n^{\frac{1}{2n}}}{n^{1/n}}|$. The denominator converges to 1, and the top is less than one, hence this α is also less than 1, therefore it converges by the root test.

7 Rudin 3.9

$$\alpha = \limsup |c_n|^{1/n}, R = \frac{1}{\alpha}$$

1.
$$\sum n^3 z^n$$
$$\alpha = \limsup |n^3|^{1/n} = \limsup |n^{3/n}| = 1 \implies R = 1$$

2.
$$\sum \frac{2^n}{n!} z^n$$
$$\alpha = \limsup |\frac{2^n}{n!}|^{1/n} = \limsup |\frac{2}{n!^{1/n}}| = 0 \implies R = \infty$$

3.
$$\sum \frac{2^n}{n^2} z^n$$
$$\alpha = \limsup |\frac{2^n}{n^2}|^{1/n} = \limsup |\frac{2}{n^{2/n}}| = \frac{2}{1} \implies R = \frac{1}{2}$$

4.
$$\sum \frac{n^3}{3^n} z^n$$
$$\alpha = \limsup |\frac{n^3}{3^n}|^{1/n} = \limsup |\frac{n^{3/n}}{3}| = \frac{1}{3} \implies R = 3$$

8 Rudin 3.11

1.
$$\lim \frac{a_n}{1 + a_n} = \lim \frac{1}{1 + \frac{1}{a_n}} \to 1 \neq 0 \text{ and fails sanity check}$$

2.
$$\begin{aligned}
\frac{a_{N+1}}{s_{N+1}} + ... + \frac{a_{N+k}}{s_{N+k}} &\geq \frac{a_{N+1}}{s_{N+k}} + ... + \frac{a_{N+k}}{s_{N+k}} \\
&\text{since } a_n > 0 \implies s_n > s_m \text{ for } n > m \\
&= \frac{s_{N+k} - s_N}{s_{N+k}} \\
&= 1 - \frac{s_N}{s_{N+k}}
\end{aligned}$$

Therefore the partial sum does not converge to zero since $\sum a_n/s_n$ will converge to 1 as $k \to \infty$ and it diverges.

3.