Math 104 Homework 3

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1 Ross 10.6

1. Consider the sequence $(U_n) = (\sum_{i=0}^{n} 2^{-n})_{i=0}^{n}$. Note that for a given n, U_n represents an upper bound on the maximum distance between any pair of points between indicies 0 and n, since it is a sum of the upperbounds between each individual consecutive pair of distances. We can rewrite this sequence as $U_n = 2 - \frac{1}{2^n}$. Also note that this sequence converges to 2. Now consider the sequence $2 - U_n$. Since U_n represents the maximum distance between any two points between indices 0 and n, the sequence represents the complement of this and is the maximum distance between any two points between indices n and any index larger than n (since we know that the maximum possible distance is 2). Hence to prove that this is cauchy, all we need to do is consider the fact that $\frac{1}{2^N}$ is an upperbound on the distance between any two points s_n, s_m where $n, m > N$. Thus for any $\epsilon > 0$, there exists such an $N = -\log_2(\epsilon)$ such that for any $n, m > N$, $|s_n - s_n| < \epsilon$ since it is an upperbound on the distances.

2. It does not satisfy the conditions for a Cauchy sequence because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to positive infinity. Hence for any $\epsilon > 0$, there is no such N such that any $n, m > N$, $|s_n - s_m| < \epsilon$, since if we take an arbitrarily large m, summing over the intermediate distances can become arbitrarily large, since we know that the total sum does not converge.

2 Ross 11.2

1. (a) (-1, -1, -1 ...) or (1, 1, 1 ...)
   (b) b_n is a decreasing sequence so any subsequence is monotonic
   (c) c_n is an increasing sequence so any subsequence is monotonic
   (d) d_n is a decreasing sequence for $n \in \mathbb{N}$ so any subsequence is monotonic.
2. (a) S = {-1, 1 }
   (b) S = 0

(c) S = $+\infty$
(d) S = 6/7

3. (a) limsup = 1, liminf = -1
   (b) limsup = 0, liminf = 0
   (c) limsup = $+\infty$, liminf = $+\infty$
   (d) limsup = 6/7, liminf = 6/7

4. (a) Does not converge nor diverge, oscillates
   (b) Converges to 0
   (c) Diverges to $+\infty$
   (d) Converges to 6/7

5. Only a_n, b_n, d_n are bounded.

3 Ross 11.3

1. (a) s_{6n} for $n = 1, 2, ...$
   (b) t_n is a decreasing sequence so any subsequence is monotonic
   (c) u_{2n} for $n = 1, 2, ...$
   (d) v_{2n} for $n = 1, 2, ...$

2. (a) S = {-1, 1, -0.5, 0.5 }
   (b) S = 0
   (c) S = $+\infty$
   (d) S = -1, 1

3. (a) limsup = 1, liminf = -1
   (b) limsup = 0, liminf = 0
   (c) limsup = 0, liminf = 0
   (d) limsup = 1, liminf = -1

4. (a) Does not converge nor diverge, oscillates
   (b) Converges to 0
   (c) Converges to 0
   (d) Oscillates but also shrinks as at oscilates.

5. All are bounded.

4 Ross 11.5

1. By the denseness of $\mathbb{Q}$, we know that there are infinitely many rationals between any $a, b \in \mathbb{R}$, in this case, we let $a = 0, b = 1$. Since there are infinitely many rationals, we know that for any rational $q \in (0, 1]$, there will exist a monotonic subsequence with the limit q, where q is the sup of the subsequence. This is because we know that we visit all the rationals between 0 and 1, and we know that there are infinitely many rationals. As a result, for any q_n, we can find a q_m such that $q_n < q_m < q$. If q_m has already been visited in the sequence, then we continue to keep looking until we find one that has not been visited. We know we will find one because we visit all rationals, and there are infinitely many rationals. As a result, and q can be a subsequential limit between (0,1], hence the set of all subsequential limits is all rational numbers between (0,1].

   EDIT: Actually, it is all real numbers between (0,1], since we know that the denseness property have real numbers as bounds, therefore there are infinitely many rational numbers that can get arbitrarily close to any real number between (0,1]. (Thanks Jonathan)

2. Since there are infinitely many rationals, at the limit of q_n, there will still be an infinitely many number of rationals that are arbitrarily close to 1 that satisfy the ϵ bound for any $\epsilon > 0$. Similarly, the lim inf is 0, as there is an infinite amount of rational numbers that can get arbitrarily close to 0.

5 Discussion on Lim Sup

1. I would explain limsup as cutting off everything before a certain index and then looking at the biggest thing for only the stuff after the cutoff. Sup looks for the least upper bound for everything while limsup only looks at it after a certain point.

2. The most counter intuitive thing about limsup is that it might not be the same magnitude as sup. if you have a monotonically decreasing sequence, the sup is the first value which might be super large, but the lim sup is going to be super tiny.