Math 104 Homework 5

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1 Ross 13.3

1. If $x = y$, then $x_j = y_j$ for $j = 1, 2, ...$ and $d(x, y) = \sup\{|x_j - y_j| = 0 : j = 1, 2, ...\} = 0$. In addition, if $x \neq y$ then there exists at least on index in which $|x_j - y_j| > 0$. Hence $d(x, y) > 0$ for distinct x, y since the supremum of absolute values will be positive.

   Next, it is readily apparent that $d(x, y) = d(y, x)$ as we just flip the order of the difference, however since it is in absolute terms, it will make no difference.

   Lastly, to prove the triangle inequality, for a third sequence $z = (z_1, z_2, ...)$ we can use the triangle inequality for each term to prove that the metric satisfies the total triangle inequality. We know that $|x_j - z_j| \leq |x_j - y_j| + |y_j - z_j|$ for all $j = 1, 2, ...$ Therefore, considering the supremum of both sides, $\sup |x_j - z_j| \leq \sup\{|x_j - y_j| + |y_j - z_j|\}$. The supremum of the sum is necessarily less than or equal to the sum of the supremums, since in this case we are considering the same index while in the latter we consider the upper bound over all indices. Hence $\sup |x_j - z_j| \leq \sup |x_j - y_j| + \sup |y_j - z_j|$. By definition, this implies that $d(x, z) \leq d(x, y) + d(y, z)$

2. No because distance functions should not go to ∞. Consider any two sequences that are bounded and converge to different number. These two sequences will have $d^*(x, y) = \infty$ since $\lim_{j\to\infty} |x_j - y_j| = |\lim_{j\to\infty} x_j - \lim_{j\to\infty} y_j| = c > 0$, hence $\sum_{j=1}^{\infty} |x_j - y_j| < \sum_{j=1}^{\infty} c = \infty$ hence the distance would go to infinity.

2 Ross 13.5

1. We wish to verify

$$\bigcap\{S \setminus U : U \in \mathcal{U}\} = S \setminus \bigcap\{U : U \in \mathcal{U}\}$$

   It suffices to reason that all points on the left hand side are the same as the points on the right hand side. On the left hand side, we are taking the intersection of all the left over bits after removing set $U \in \mathcal{U}$ from S

and on the right hand side, we are figuring out the total union of $\mathcal{U}$ before removing it from S. If a point is in the left hand side, that means it was not in U for all $\mathcal{U}$. Similarly, if it is not in U for all $\mathcal{U}$, then it will also not be in the union of U. Hence all points in the left are equal to the points on the right.

2. Using the above equality, if we consider U to be some set of open sets, then we know that $\bigcap\{S \setminus U : U \in \mathcal{U}\}$ is the intersection of a collection of closed sets by definition 13.8 which states that E is closed if $E = S \setminus U$ where U is an open set. Hence, we know that the intersection of a collection of closed sets is equal to the difference between S and the intersection of a bunch of open sets. We know that the intersection of a collection of open sets is an open set and hence by definition, the right hand side is the difference between a set and an open set and hence is a closed set.

3 Ross 13.7

We want to show that every open set in $\mathbb{R}$ is a disjoint union of a finite or infinite sequence of open intervals. By definition, every point x in an open set lies in the interior of S which means for some radius $r > 0$, all points $\{y : d(x, y) < r\}$ also lie in S. If we consider the open interval which covers this radius for every point x, then the union of all these open intervals will form $S \in \mathbb{R}$. However, we want to prove that each open set is the *disjoint* union, and currently these open intervals might overlap. To make these sets disjoint, we can recursively merge together sets that overlap until the remaining sets are disjoint, and hence the open set in $\mathbb{R}$ is a disjoint union of open intervals.

4 Closure of a Closure

We want to prove that taking the closure of a closure gives you the same thing. The closure of S is defined as $S^- = \{p \in X \mid \exists (p_n) \in S \text{ s.t. } \lim(p_n) = p\}$. We would like to show that for every point in S^-, there is a subsequence in S^- that converges to that point, and hence its closure is equal to itself. Based on the definition of S^-, we know that there exists a sequence $(p_n) \in S$ that converges to a point s in S^-. Hence, by using Cantor's diagonalization trick, we can show that there exists a sequence in S^- that converges s. Since the closure is the intersection of all closed sets, it is necessarily closed, and hence we know that there will be a sequence $(x_j) \in S^-$ for $j = 1, 2, ...$ that is $\pm\epsilon/j$ away from s for any $\epsilon > 0$. Since we can arbitrarily decrease how close our sequence is around s, this if we pick the new sequence $(x_j)_j)$ for $j = 1, 2, ...$ then it will converge to s. Since s is a limit in S^- and this holds for any s in S^-, then the closure of $S^- = S^-$

5 Intersection of Closed Subsets

Suppose we have an arbitrary $x \in S^-$ where S^- is the closure of S which is some subset of X. Let C be a closed subset of X such that $S \subseteq C$. Suppose that $x \notin C$, then that means that x is also not in S. However, we also know that $S^- \subseteq S$, hence this is a contradiction, and therefore $x \in C$ for any arbitrary C that contains S. Since this must hold for any arbitrary C, then it must be the case that S^- is the intersection of all these closed sets.