Ross 1.10✓, 1.12✓, 2.1✓, 2.2✓, 2.7✓, 3.6✓, 4.11✓, 4.14✓, 7.5✓

1.10) Prove $(2n+1)+(2n+3)+\dots+(4n-1)=3n^2$ for all positive integers n.

Base Case: $n=1$ $\quad (2(1)+1) = 3(1)^2$ ✓

Assume that $(2n+1)+(2n+3)+(2n+5)+\dots+(4n-1)=3n^2$

$$(2(n+1)+1)+(2(n+1)+3)+(2(n+1)+5)+\dots+(4(n+1)-1)$$

$$= \underbrace{(2n+3)+(2n+5)+(2n+7)+\dots+4n-1}_{=3n^2-(2n+1) \text{ by inductive hypothesis}} + (4n+1)+(4n+3)$$

$$= 3n^2-2n-1+4n+1+4n+3$$

$$= 3n^2+6n+3$$

$$= 3(n+1)^2 \quad \square$$

1.12) a.) Verify binomial theorem for $n=1,2,3$

$n=1$: $\quad (a+b)^1 = \binom{1}{0}a^1 + \binom{1}{1}b^1$

$= a+b$ ✓

$n=2$: $\quad (a+b)^2 = \binom{2}{0}a^2 + \binom{2}{1}a^1b^1 + \binom{2}{2}b^2$

$= a^2+2ab+b^2$ ✓

$n=3$: $\quad (a+b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$

$= a^3+3a^2b+3ab^2+b^3$ ✓

b.) Show $\binom{n}{k}+\binom{n}{k-1}=\binom{n+1}{k}$ for $k=1,2,\dots,n$

$$\frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}$$

$$= \frac{n!(n-k+1)}{k!(n-k+1)!} + \frac{n!\,k}{k!(n-k+1)!}$$

$$= \frac{n!(n-k+1+k)}{k!(n-k+1)!}$$

$$= \frac{n!(n+1)}{k!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}$$ ✓

c.) Base Case: part a covers it

Assume $(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$

$$(a+b)^{n+1} = (a+b)\left[\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n\right]$$

$$= \binom{n}{0}a^{n+1} + \binom{n}{0}a^nb + \binom{n}{1}a^nb + \binom{n}{1}a^{n-1}b^2 + \dots + \binom{n}{n}ab^n + \binom{n}{n}b^{n+1}$$

$$= \binom{n}{0}a^{n+1} + \left[\binom{n}{0}+\binom{n}{1}\right]a^nb + \dots + \left[\binom{n}{n-1}+\binom{n}{n}\right]ab^n + \binom{n}{n}b^{n+1}$$

$= \binom{n}{0}a^{n+1} + \binom{n+1}{1}a^nb + \dots + \binom{n+1}{n}ab^n + \binom{n}{n}b^{n+1}$ by inductive hypothesis

$$= \binom{n+1}{0}a^{n+1} + \binom{n+1}{1}a^nb + \dots + \binom{n+1}{n}ab^n + \binom{n+1}{n+1}b^{n+1} \quad \square$$

2.1) Show $\sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{24}, \sqrt{31}$ are not rational numbers

Use the rational roots theorem.

$\sqrt{3}$: $x^2-3=0$ has possible solutions $\pm1, \pm3$, neither work

$\sqrt{5}$: $x^2-5=0$ has possible solutions $\pm1, \pm5$, neither work

$\sqrt{7}$: $x^2-7=0$ has possible solutions $\pm1, \pm7$, neither work

$\sqrt{24}$: $x^2-24=0$ has possible solutions $\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$ and none work

$\sqrt{31}$: $x^2-31=0$ has possible solutions $\pm1, \pm31$, neither work

2.2) Show $\sqrt[3]{2}, \sqrt[7]{5}, \sqrt[4]{13}$ are not rational numbers

Use the rational roots theorem

$\sqrt[3]{2}$: $x^3-2=0$ has possible solutions $\pm1, \pm2$ neither work

$\sqrt[7]{5}$: $x^7-5=0$ has possible solutions $\pm1, \pm5$ neither work

$\sqrt[4]{13}$: $x^4-13=0$ has possible solutions $\pm1, \pm13$ neither work

2.7) Show the following are rational

a.) $\sqrt{4+2\sqrt{3}}-\sqrt{3}$

$x = \sqrt{4+2\sqrt{3}}-\sqrt{3}$

$x+\sqrt{3} = \sqrt{4+2\sqrt{3}}$

$x^2+2\sqrt{3}x+3 = 4+2\sqrt{3}$

$x^2+2\sqrt{3}x-1-2\sqrt{3}=0$

$x=1$ is a solution to this equation and 1 is rational.

b.) $\sqrt{6+4\sqrt{2}}-\sqrt{2}$

$x = \sqrt{6+4\sqrt{2}}-\sqrt{2}$

$x+\sqrt{2} = \sqrt{6+4\sqrt{2}}$

$x^2+2\sqrt{2}x+2 = 6+4\sqrt{2}$

$x^2+2\sqrt{2}x-4-4\sqrt{2}=0$

$x=2$ is a solution which is rational

3.6) a.) Prove $|a+b+c| \le |a|+|b|+|c| \quad \forall a,b,c \in \mathbb{R}$

Triangle inequality:

$|a+b+c| \le |a| + |b+c|$

$|a|+|b+c| \le |a|+|b|+|c|$

$\Rightarrow |a+b+c| \le |a|+|b|+|c|$

b.) Prove $|a_1+a_2+\dots+a_n| \le |a_1|+|a_2|+\dots+|a_n|$

Base case: For $n=2$, this is given by the triangle inequality

Assume $|a_1+a_2+\dots+a_n| \le |a_1|+\dots+|a_n|$

Then, $|a_1+\dots+a_{n+1}| \le \underbrace{|a_1+\dots+a_n|}_{\le |a_1|+\dots+|a_n| \text{ from inductive hypothesis}} + |a_{n+1}|$

by the triangle inequality

$|a_1+\dots+a_{n+1}| \le |a_1|+\dots+|a_n|+|a_{n+1}|$

which is the desired inequality □

4.11) $a,b \in \mathbb{R}$ $a<b$. Show infinitely many rationals between a and b.

The denseness of $\mathbb{Q}$ (4.7) says there exists a rational $r \in \mathbb{Q}$ s.t. $a<r<b$. Intuitively, we can find infinitely many smaller rationals by applying the denseness statement recursively.

(Meta: To prove it more formally, I'm hesitant to use induction (since we're dealing with infinity). I'll do contradiction.)

Assume there are a finite number of rationals between a and b. Take the maximum and call it r. By the denseness of $\mathbb{Q}$, $\exists r'$ s.t. $r<r'<b$, but this makes r' the new maximum, which is a contradiction.
Therefore, there are an infinite number of rationals between a and b.

4.14) a.) Prove $\sup(A+B) = \sup(A)+\sup(B)$

$\rightarrow$: $\sup(A+B) \le \sup(A)+\sup(B)$

let $c \in A+B$. $c=a+b$ for some $a \in A$ and $b \in B$. Therefore $c \le \sup(A)+\sup(B)$

and because c is arbitrary, $\sup(A+B) \le \sup(A)+\sup(B)$.

$\leftarrow$ $\sup(A+B) \ge \sup(A)+\sup(B)$

let $c \in A+B$. $c=a+b$ $a \in A$, $b \in B$

$a+b \le \sup(A+B)$

$a \le \sup(A+B)-b$

so for all possible b's,

$\sup(A) \le \sup(A+B)-b$ aka

$\sup(A+B)-b$ is an upper bound for A

$b \le \sup(A+B)-\sup(A) \quad \forall b \in B$

$\sup(B) \le \sup(A+B)-\sup(A)$

$\sup(A)+\sup(B) \le \sup(A+B)$

$\Rightarrow \sup(A)+\sup(B) = \sup(A+B)$

b.) Prove $\inf(A+B) = \inf(A)+\inf(B)$

$\rightarrow$ $\inf(A+B) \ge \inf(A)+\inf(B)$

let $c \in A+B$ $c=a+b$ $a \in A$ $b \in B$

$c=a+b \ge \inf(A)+\inf(B)$ for all possible a and b, so $\inf(A+B) \ge \inf(A)+\inf(B)$

$\leftarrow$ $\inf(A+B) \le \inf(A)+\inf(B)$

let $c \in A+B$ $c=a+b$ $a \in A$ $b \in B$

$a+b \ge \inf(A+B)$

$a \ge \inf(A+B)-b$

$\forall b \in B$, $\inf(A) \ge \inf(A+B)-b$

$b \ge \underbrace{\inf(A+B)-\inf(A)}_{\text{lower bound for } B}$

$\inf(B) \ge \inf(A+B)-\inf(A)$

$\inf(B)+\inf(A) \ge \inf(A+B)$

$\Rightarrow \inf(A+B) = \inf(A)+\inf(B)$

7.5) a.) $\lim s_n$ $\quad s_n = \sqrt{n^2+1} - n$

$$\sqrt{n^2+1} - n \cdot \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n}$$

$$= \frac{1}{\sqrt{n^2+1}+n} \qquad \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}+n} = 0$$

b.) $\lim (\sqrt{n^2+n} - n)$

$$\lim_{n\to\infty} \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$$

c.) $\lim (\sqrt{4n^2+n} - 2n)$

$$\lim_{n\to\infty} \frac{4n^2+n-4n^2}{\sqrt{4n^2+n}+2n} = \lim_{n\to\infty} \frac{n}{\sqrt{4n^2+n}+2n}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{4+\frac{1}{n}}+2} = \frac{1}{4}$$