9.9✓, 9.15✓, 10.7✓, 10.8✓

10.9✓, 10.10✓, 10.11✓

Sqeeze test✓

9.9) $\exists N_0$ s.t. $s_n \le t_n \quad \forall n > N_0$

a.) Prove if $\lim s_n = +\infty$, then $\lim t_n = +\infty$

Using the definition of 'diverging to $+\infty$' for s_n:

For any $M \in \mathbb{R}$, we know $\exists N$ s.t. $s_n > M$ for all $n > N$.

Call $N' = \max\{N, N_0\}$ (forcing N' to be $\ge$ than N and N_0)

Then, for all $n > N'$, $t_n \ge s_n > M$, so $\lim t_n = +\infty$.

b.) Prove if $\lim t_n = -\infty$, then $\lim s_n = -\infty$

For any $M \in \mathbb{R}$, $\exists N$ s.t. $t_n < M \quad \forall n > N$.

Call $N' = \max\{N, N_0\}$

$\forall n > N'$, $s_n \le t_n < M \Rightarrow \lim s_n = -\infty$

c.) Prove if $\lim s_n$ and $\lim t_n$ exist, then $\underbrace{\lim s_n}_{a} \le \underbrace{\lim t_n}_{b}$

Call $\lim s_n = a$ and $\lim t_n = b$

then $\lim(s_n - t_n) = a - b$ (proof in lecture)

$\Rightarrow \forall \varepsilon \; \exists N$ s.t. $\forall n > N \; |(t_n - s_n) - (a-b)| < \varepsilon$

Proof by contradiction: Assume $a > b$

Call $N' = \max\{N, N_0\}$

We know $\forall n > N'$: $s_n \le t_n$ but $a > b$

$\Rightarrow \underbrace{(t_n - s_n)}_{\text{positive quantity}} - \underbrace{(b-a)}_{\text{negative quantity}} > 0$

This is a contradiction since you can just pick an ε smaller than $(t_n - s_n) - (b-a)$ (which is some positive number).

9.15) Show $\lim_{n\to\infty} \frac{a^n}{n!} = 0 \quad \forall a \in \mathbb{R}$

$$\frac{a^n}{n!} = \frac{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n}$$

$$= \underbrace{\frac{a \cdot a \cdot a \cdots a}{1 \cdot 2 \cdots (a-1) \cdot a}}_{\text{treat this as some number}} \; \underbrace{\frac{a \cdot a \cdot a \cdot}{(a+1) \cdots n}}$$

$$= \lim_{n\to\infty} \frac{a}{a+1} \cdot \frac{a}{a+2} \cdots \frac{a}{n}$$

$$= 0$$

10.7) Use squeeze thm to construct s_n

$(t_n) = \sup S$

$(u_n) = \sup S - \frac{1}{n}$ goes to 0

We know $\forall \varepsilon > 0 \; \exists s' \in S$ s.t. $s' > \sup S - \varepsilon$ (definition)

so construct (s_n) as all elements S where $s_n > \sup S - \frac{1}{n}$. It will be nonempty because consider $\varepsilon = \frac{1}{n}$.

$t_n \ge s_n \ge u_n \Rightarrow (s_n) \to \sup S$

10.8)

$$\sigma_{n+1} = \frac{1}{n+1}(s_1 + \ldots + s_{n+1})$$

$$= \frac{1}{n+1}(n\sigma_n + \underbrace{s_{n+1}}_{\ge \sigma_n})$$

$$\left(\begin{array}{l} s_{n+1} \ge \text{all previous } s_n\text{'s} \\ n \cdot s_{n+1} \ge s_1 + s_2 + \ldots + s_n \\ s_{n+1} \ge \sigma_n \end{array} \right)$$

$$\ge \frac{1}{n+1}(n\sigma_n + \sigma_n)$$

$$= \frac{1}{n+1}((n+1)\sigma_n)$$

$$= \sigma_n$$

so $\sigma_{n+1} \ge \sigma_n$

10.9) a.) $s_2 = \frac{1}{2}(1)^2 = \frac{1}{2}$

$s_3 = \frac{2}{3}(\frac{1}{2})^2 = \frac{1}{6}$

$s_4 = \frac{3}{4}(\frac{1}{6})^2 = \frac{1}{48}$

b.) We see that $s_{n+1} = \underset{<1}{\underset{\uparrow}{\frac{n}{n+1}}} \underset{<1}{\underset{\uparrow}{(s_n)^2}}$

for $n \geq 2$. This means the sequence is decreasing but also bounded since $\frac{n}{n+1}(s_n)^2 \geq 0$ (and so is bounded by 0 below.

$\Rightarrow$ a limit exists

c.) $(t_n) = 0$

$(u_n) = \frac{1}{n}$

Prove $s_n \leq \frac{1}{n}$:

Base case: $1 \leq 1$ ✓

Inductive step: $s_{n+1} = \frac{n}{n+1}(s_n)^2 \leq \frac{n}{n+1}(\frac{1}{n})^2 = \frac{1}{n(n+1)} \leq \frac{1}{n+1}$

$$\underset{\lim t_n = 0}{\underset{\uparrow}{t_n}} \leq s_n \leq \underset{\lim u_n = 0}{\underset{\uparrow}{u_n}}$$

$\Rightarrow \lim s_n = 0$

10.10)

a.) $s_2 = \frac{1}{3}(1+1) = \frac{2}{3}$

$s_3 = \frac{1}{3}(\frac{2}{3} + \frac{3}{3}) = \frac{5}{9}$

$s_4 = \frac{1}{3}(\frac{5}{9} + \frac{9}{9}) = \frac{14}{27}$

b.) $s_n > \frac{1}{2}$

Base Case: $s_1 = 1 > \frac{1}{2}$ ✓

$s_{n+1} = \frac{1}{3}(s_n + 1) > \frac{1}{3}(\frac{1}{2} + 1) = \frac{1}{2} \quad \Rightarrow s_n > \frac{1}{2}$

c.) WTS: $s_{n+1} < s_n$

$s_{n+1} = \frac{1}{3}(s_n + 1) < \frac{1}{3}(s_n + 2s_n) = s_n$

since $s_n > \frac{1}{2}$

$\Rightarrow s_{n+1} < s_n$

d.) $\lim s_{n+1} = \lim \frac{1}{3}(s_n + 1) = s$

$s = \frac{1}{3}s + 1$

$s = \frac{1}{2}$

10.11) a.) Show bounded and decreasing

Bounded below by 0:

Induction on $t_n > 0$

Base case: $t_1 = 1 > 0$

$t_{n+1} = \underbrace{\left(1 - \frac{1}{4(n+1)^2}\right)}_{>0 \text{ for } n \geq 1} t_n$ ↙ >0 by inductive hypothesis

$\Rightarrow t_{n+1} > 0$

Decreasing: WTS $t_{n+1} \leq t_n$

$(1 + \frac{1}{4n^2})t_n = \frac{1}{4n^2}(4n^2 + 1)t_n$

$t_{n+1} = \underbrace{\frac{4n^2+1}{4n^2}}_{>1} t_n$

$t_{n+1} \leq t_n$ ✓

b.) 0? edit: it is not 0 :(

Squeeze Test:

$a_n \leq b_n \leq c_n \quad L = \lim a_n = \lim c_n$

Show $\lim b_n = L$

$\forall \varepsilon > 0 \quad \exists N_0$ s.t. $\forall n > N_0 \ |a_n - L| < \varepsilon$

$\exists N_1$ s.t. $\forall n > N_1 \ |c_n - L| < \varepsilon$

Set $N = \max\{N_0, N_1\}$

$\forall n > N$:

$L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$

$L - \varepsilon < b_n < L + \varepsilon$

$|b_n - L| < \varepsilon$

$\lim b_n = L$