HW 3

Ross 10.6 ✓ 11.2 ✓ 11.3 ✓ 11.5 ✓

limsup questions

10.6) a.) $|s_{n+1}-s_n| < 2^{-n} \quad \forall n \in \mathbb{N}$

WTS: $\forall \varepsilon > 0 \ \exists N > 0$ s.t. $\forall n,m > N$,

WLOG $n>m$ $\quad |s_n - s_m| < \varepsilon$

$|s_n - s_m| = |s_n - s_{n-1} + s_{n-1} - s_{n-2} + s_{n-2} + \dots + s_{m+1} - s_m|$

$\leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \dots + |s_{m+1} - s_m|$

$< \underbrace{2^{-n} + 2^{-n+1} + \dots 2^{-m-1}}_{< 2^{-n}}$ (ex: $\frac{1}{2} > \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$ binary counting)

$< 2^{-n+1}$

we can then find an n big enough such that

$2^{-n+1} < \varepsilon \qquad -n+1 < \log_2 \varepsilon$

$-n < \log_2 \varepsilon - 1$

$n > 1 - \log_2 \varepsilon$

b.) no because

$\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \dots$ doesn't converge

11.2) $a_n = (-1)^n \quad b_n = \frac{1}{n} \quad c_n = n^2 \quad d_n = \frac{6n+4}{7n-3}$

a.) $a_1, a_3, a_5, a_7 \dots \quad \{a_{2k} \mid k \in \mathbb{Z}, k>0\}$

$b_1, b_2, b_3, b_4 \dots \quad \{b_k \mid k \in \mathbb{Z}, k>0\}$

$c_1, c_2, c_3, c_4 \dots \quad \{c_k \mid k \in \mathbb{Z}, k>0\}$

$d_1, d_2, d_3, d_4 \dots \quad \{d_k \mid k \in \mathbb{Z}, k>0\}$

b.) $a_n: \{-1, 1\}$

$b_n: \{0\}$

$c_n: \{+\infty\}$

$d_n: \{6/7\}$

c.) $\limsup a_n = 1 \quad \liminf a_n = -1$

$\limsup b_n = 0 \quad \liminf b_n = 0$

$\limsup c_n = +\infty \quad \liminf c_n = +\infty$

$\limsup d_n = \frac{6}{7} \quad \liminf d_n = \frac{6}{7}$

d.) $b_n, d_n \to$ converge

$c_n \to$ diverges to $+\infty$

$a_n \to$ oscillates

e.) a_n, b_n, d_n

11.3) $s_n = \cos\left(\frac{n\pi}{3}\right) \quad t_n = \frac{3}{4n+1} \quad u_n = \left(\frac{-1}{2}\right)^n \quad v_n = (-1)^n + \frac{1}{n}$

a.) $s_3, s_9, s_{15}, \dots \quad \{s_{6k} \mid k \in \mathbb{Z}^+\}$

$t_1, t_2, t_3, \dots \quad \{t_k \mid k \in \mathbb{Z}^+\}$

$u_1, u_3, u_5, \dots \quad \{u_{2k} \mid k \in \mathbb{Z}^+\}$

$v_1, v_3, v_5, \dots \quad \{v_{2k} \mid k \in \mathbb{Z}^+\}$

b.) $s_n: \{1, -1, 1/2, -1/2\}$

$t_n: \{0\}$

$u_n: \{0\}$

$v_n: \{-1, 1\}$

c.) $\limsup s_n = 1 \quad \liminf s_n = -1$

$\limsup t_n = 0 \quad \liminf t_n = 0$

$\limsup u_n = 0 \quad \liminf u_n = 0$

$\limsup v_n = 1 \quad \liminf v_n = -1$

d.) $t_n, u_n \to$ converge

$v_n, s_n \to$ oscillate

e.) s_n, t_n, u_n, v_n

11.5) a.) reals in $[0,1]$

from denseness of rationals, there are infinitely many rationals between any 2 reals.

$r - \varepsilon < q < r \leftarrow r \in \mathbb{R}$,

$\uparrow \forall \varepsilon > 0$

infinitely many $q \in \mathbb{Q}$.

$\Rightarrow \{q \mid |q-r| < \varepsilon\}$ is an infinite set for any ε and any r.

$\Rightarrow$ all r's are subsequential limits.

b.) $\limsup q_n = 1$

$\liminf q_n = 0$

$$\limsup_{n\to\infty} a_n = \lim_{N\to\infty}\Big(\underbrace{\sup \underbrace{\{a_n : n \geq N\}}_{\text{set}}}_{\#}\Big)$$

In English, limsup is the limit of the sup of the tail of a sequence as you move further and further down the tail (increasing N).

I think the most unintuitive thing about limsup is that it can be less than the sup of a sequence