January 18^{th}

homework submission: upload to student homepage and then share to discord

also gradescope (grader feedback)

$\mathbb{N}, \mathbb{Q}, \mathbb{R}$

Natural numbers

$\mathbb{N} = \{0, 1, 2, \dots\}$

- Successor construction: 2 is a successor of 1

  3 ——— of 2

  so starting from 0, one can reach all the naturals

- Peano Axioms for natural numbers (Tao - I)

  mathematical induction property (Axiom 5):

  → Let $n \in \mathbb{N}$. $P(n)$ be a statement depending on n. If (1), (2) hold,

  (1) $P(0)$

  (2) $P(k) \Rightarrow P(k+1)$

  then $P(n)$ is true for all $n \in \mathbb{N}$

- "operations allowed for $\mathbb{N}$": $+, \times$ (closure)

  ($-$, $/$ not always defined)

Integers

- $\mathbb{Z} = \{\dots -2, -1, 0, 1, 2, \dots\}$

  allowed operation: $+, -, \cdot$

  (formally $\mathbb{Z}$ is a ring) (Harrison Chen's note)

Rational Numbers

- $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$
- all 4 operations

  $\mathbb{Q}$ is now a "field"

  → $\mathbb{Q}$ is an ordered field, there is a "relation" $\leq$.

⌈A relation S is a subset of $\mathbb{Q} \times \mathbb{Q}$, if $(a,b) \in S$, we say "a and b has relation S" or "a S b"⌋

The "$\leq$" relation has 3 properties

① $a \leq b, b \leq a \Rightarrow a = b$

② $a \leq b, b \leq c \Rightarrow a \leq c$ (transitivity)

③ $\forall a, b \in \mathbb{Q}$, $a \leq b$ OR $b \leq a$ OR both

Ordered Field $\mathbb{Q}$: field structure $(+, -, \cdot, /)$ is compatible w/ relation structure $(\leq)$

(a) $a \leq b \Rightarrow a + c \leq b + c$, $c \in \mathbb{Q}$

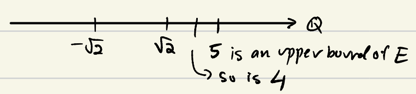
(b) $a \geq 0, b \geq 0 \Rightarrow a \cdot b \geq 0$

What's lacking about $\mathbb{Q}$?

There are certain gaps in $\mathbb{Q}$: for example, the equation $x^2 = 2$ can't be solved in $\mathbb{Q}$.

For a bounded subset in $\mathbb{Q}$ called E, it may not have a "most economical" or "sharpest" upper bound in $\mathbb{Q}$

ex: $E = \{x \in \mathbb{Q} \mid x^2 \leq 2\}$



but $\underbrace{\text{no least upper bound}}_{\sup(E)}$

↳ we wanna say $\sqrt{2}$ but $\sqrt{2} \notin \mathbb{Q}$