Math 104

0. 1. How do we keep an eye out for the Root test besides finding exponents to nth power?
2. When do we tend to use summation of parts in problems? Seems very niche
3. What is the use if knowing a series is absolutely convergent.
4. For the integral test, do need to assume the integral is $\geq$ or $\leq$ to the series
5. What was the difference in content between chp. 12 and the past chapter on limsup/inf

12.10)
1. Prove (s_n) bounded iff $\limsup |s_n| < +\infty$

1) if $\limsup |s_n| < +\infty$, means for $s = \limsup |s_n| < \infty$, that $\forall \epsilon > 0$, $\exists N$ that $\forall n > N$ $s - \epsilon < \sup |s_n| < s + \epsilon \to -s - \epsilon < \sup s_n < s + \epsilon$, so by def of sup, all s_n must be $-\infty < s_n < +\infty$, so (s_n) is bounded

2) If (s_n) is bounded, for all $n \to |s_n| < +\infty$, so there exists some $0 \leq s < +\infty$ that $|s_n| < s$, so $\limsup |s_n| \leq s < +\infty$ ✓

14.2)

a) $\sum \frac{n-1}{n^2} = \sum \frac{1}{n} - \frac{1}{n^2} = \sum \frac{1}{n} - \sum \frac{1}{n^2}$ = divergent since $\sum \frac{1}{n}$ diverges bc harmonic seq, $\sum \frac{1}{n^2}$ converges from Integral Test, so it's a constant

b) $\sum (-1)^n \to$ not Cauchy since for all $n > 1$, $|s_{n+1} - s_n| = 1$, so not for all $\varepsilon > 0$ will there exist a N that $\forall n > N$ $|s_{n+1} - s_n| < \varepsilon$

c) $\sum \frac{3n}{n^2} = \sum \frac{3}{n} = 3 \cdot \sum \frac{1}{n} \Rightarrow$ divergent bc harmonic

d) $\sum \frac{n^3}{3^n}$ Ratio test: $\left| \frac{(n+1)^3}{3^{(n+1)}} \cdot \frac{3^n}{(n)^3} \right| = \left| \frac{(n+1)^3}{3n^3} \right| \to 0$ as $n \to \infty$, so it is absolutely convergent

e) $\sum \frac{n^2}{n!}$ Ratio test: $\left| \frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2} \right| = \left| \frac{(n+1)^2}{n^2} \cdot \frac{1}{n+1} \right| = \left| \frac{n+1}{n^2} \right| \to 0$ as $n \to \infty$ so absolutely conv

f) $\sum \frac{1}{n^n}$ Root test: $\lim_{n \to \infty} \sqrt[n]{\left| \frac{1}{n^n} \right|} = \lim_{n \to \infty} \left| \frac{1}{n} \right| = 0$, so absolutely convergent

g) $\sum \frac{n}{2^n}$ Ratio test: $\left| \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right| = \left| \frac{n+1}{2n} \right| = \left| \frac{1 + \frac{1}{n}}{2} \right| \to \frac{1}{2}$ as $n \to \infty$, so absolutely convergent

$|a_{n+1} - a_n| < \varepsilon$ 2^{n+1-} $\frac{\sqrt{a_m}}{n} - \frac{\sqrt{a_n}}{n}$ $2^{\cancel{0}} \cancel{2^{-2}}$

$(\quad - a_n), -(-1)^n$ $\frac{2^{(-1)^{n+1} - n - 1}}{2^{(-1)^n - n}} = 2^{(-1)^{n+1} - 1}$ $\frac{1}{2}$ 2

$\sqrt{a_{n+1}} + \sqrt{a_n}$

$-1^n (2 - 1)$

14.10. Consider $a_n = 2^{n - (-1)^n}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{2^{n+1-(-1)^{n+1}}}{2^{n-(-1)^n}} = 2^{1 + (-1)^n + (-1)^n}$

$= 2^{1 - 2(-1)^n}$

$\liminf \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{2} < 1 < 8 = \limsup \left| \frac{a_{n+1}}{a_n} \right| \to$ ratio no info

Root test: $|a_n|^{\frac{1}{n}} = 2^{(n - (-1)^n)\frac{1}{n}} = 2^{1 - \frac{1}{n}(-1)^n} = 2^1 \to$ diverges

Rudin 6. a) $a_n = \sqrt{n+1} - \sqrt{n}$

$\sum a_n = (\sqrt{2} - 1) + (\sqrt{3} - \sqrt{2}) \ldots \sqrt{n+1} - \sqrt{n} = \sqrt{n+1} - 1 \to$ diverges as $n \to \infty$

b) $\sum \frac{\sqrt{n+1} - \sqrt{n}}{n} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{n+1-n}{n(\sqrt{n+1} + \sqrt{n})} = \frac{1}{n(\sqrt{n+1} + \sqrt{n})} \longrightarrow$ diverges as $n \to \infty$

c) $a_n = (\sqrt[n]{n} - 1)^n$ $\sum a_n \to$ Root test: $|a_n|^{\frac{1}{n}} = n^{\frac{1}{n}} - 1 \to 0$ as $n \to \infty$, so absolutely converges

7. ~~$\sum \frac{\sqrt{a_n}}{n} \leq \sum \frac{a_n}{n} < \sum a_n$, a_n converges so $\sum \frac{\sqrt{a_n}}{n}$~~

Since a_n is convergent, it satisfies Cauchy criterion, so for any ε, $\exists N$ s.t. $\forall m > n > N$; $a_m - a_n < \varepsilon$

W.T.S. $\frac{\sqrt{a_m}}{m} - \frac{\sqrt{a_n}}{n} < \varepsilon$, we know $\frac{\sqrt{a_m}}{m} - \frac{\sqrt{a_n}}{n} > \frac{\sqrt{a_m}}{m} - \frac{\sqrt{a_n}}{m} = \frac{a_m - a_n}{m(\sqrt{a_m} + \sqrt{a_n})}$

$< \frac{\varepsilon}{m(\sqrt{a_m} + \sqrt{a_n})} < \frac{\varepsilon}{1} < \varepsilon \to$ so $\frac{\sqrt{a_n}}{n}$ is cauchy $\to$ convergent

$(m(\sqrt{a_m} + \sqrt{a_n}) > 1$ if we construct a large enough N from archimedian principle)

$2n$ -2^n 1 $\frac{(n)^{\frac{1}{2}}}{2}$

9. a) $\alpha = \limsup (|a_n|^{\frac{1}{n}}) = (n^{\frac{3}{n}}) = (n^{\frac{1}{n}})^3 = \boxed{1} \to \frac{1}{\alpha} = 1$

b) $\alpha = \limsup \left(\left| \frac{2^n}{n!} \right|^{\frac{1}{n}} \right) \leq \limsup \left(\left| \frac{2^{n+1}}{n+1!} \cdot \frac{n!}{2^n} \right| \right) = \limsup \left(\frac{2}{n+1} \right) = 0$, so

$\alpha = 0 \to R = \infty$

$\sigma_n = \frac{1}{n}(s_1 + \ldots s_N \ldots s_n)$

$n > M > N$

$\leq \frac{1}{n}(s_1 + s_N + \frac{n-N}{n} \sup\{n > N\})$

(12.12) a) W.T.S. $\limsup \sigma_n \leq \limsup s_n$. For $n > M > N$, we have

$$\sigma_n = \frac{1}{n}(s_1 + \ldots s_N + \ldots s_n) = \frac{1}{n}(s_1 + \ldots s_N) + \frac{1}{n}(s_{N+1} \ldots + s_n)$$

$$\leq \frac{1}{n}(s_1 \ldots + s_N) + \frac{n-N}{n}(\sup\{s_n : n > N\}) \leq \frac{1}{M}(s_1 \ldots + s_N) + \sup\{s_n : n > N\}$$

$\rightarrow \sup\{\sigma_n : n > M\} \leq \frac{1}{M}(s_1 \ldots + s_N) + \sup\{s_n : n > N\}$

as $M \to \infty \rightarrow \limsup \sigma_n \leq 0 + \limsup s_n \rightarrow$ symmetrically, we prove the 1st equality and the 2nd is trivial from definitions

b) If $\lim s_n$ exists $\rightarrow \liminf s_n = \limsup s_n \rightarrow$ by Squeeze thm, $\liminf \sigma_n = \limsup \sigma_n$, so $\lim \sigma_n$ exists and is $= \lim s_n$

c) $s_n = \sin(\frac{\pi}{2}x) + 1 \rightarrow$ oscillates so $\lim s_n$ DNE

$\sigma_n = \frac{1}{n}(s_1 + \ldots s_n) \rightarrow 0$ as $n \to \infty$ since avg of $\sin(x)$ is 0 already