d is fcn for ~~to~~

1) $d(x,x)=0$ $d(x,y)>0$
2) $d(x,y)=d(y,x)$
3) triangle ineq

ℓ ~~to~~ k

intersection of (~~d~~ differences btwn S and u)

diff btwn S ~~&~~ union of all u

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13.3 a) $d(x,y)$ sup of all differences btwn x, y

1) $d(x,x)=0$ since all $|x_j - x_j| = 0$

$d(x,y)>0$ since $\sup\{|x_j - y_j| : j=1,2..\}=0$ only if $x_j = y_j$ for all j, which would imply $x=y$, which is impossible

2) $d(x,y) = \sup\{|x_j - y_j| : j=1,2..\} = \sup\{|y_j - x_j| : j=1,2\} = d(y,x)$

3) $d(x,z) \leq d(x,y) + d(y,z)$

let i be the index where $\sup\{|x_j - y_j| : j=1,2\} = |x_i - y_i|$ and similarly k for $\sup\{|y_j - z_j| : \ldots\}$. For $d(x,z)$, we know $\exists \ell$ similarly for $d(x,z) \to \sup\{|x_j - z_j| : \ldots\} = |x_\ell - z_\ell| = |x_\ell - y_\ell + y_\ell - z_\ell| \leq |x_\ell - y_\ell| + |y_\ell - z_\ell|$, but we know i,k are the indices for the sup of $|x_j - y_j|$ & $|y_j - z_j|$ respectively, so $|x_\ell - y_\ell| + |y_\ell - z_\ell| \leq |x_i - y_i| + |y_k - z_k|$ ✓

b) $d^*(x,y) = \sum_{j=1}^{\infty} |x_j - y_j|$

No, since the distance function d^* can return non-real values like $+\infty$ since the sum of bounded infinite sequences can diverge to $+\infty$.

13.5 a) Prove: $\bigcap\{S \setminus U : U \in \mathcal{U}\} = S \setminus \bigcup\{U : U \in \mathcal{U}\}$

An x will only be $\in \bigcap\{S\setminus U : U \in \mathcal{U}\}$ iff it's not contained in any U but is in S. Similary this is true for $\in S \setminus \bigcup\{U : U \in \mathcal{U}\}$ since $\bigcup\{U : U \in \mathcal{U}\}$ represents all u in any U, so therefore the 2 sets must be equivalent.

closure $\equiv$ intersection of all closed sets containing E

E is closed if $S\setminus E$ is open

Intersection of any closed sets is closed

$u = \{u \mid u \in \mathcal{U}\}$

b) For closed sets we have $\cap u$ be their intersection. We know $S\setminus u$ must be open, so from part a $S\setminus \cap u = \cup S\setminus u$ which is the union of open sets which is open, so $S\setminus \cap u$ is an open set, so $\cap u$ must then be a closed set.

13.7. $\forall$ open sets in $\mathbb{R}$ = disjoint union of finite seq of open intervals (Show U is disjoint union, show it has to be countable) - union of all intervals containing p but contained to U ↳ can only exist 1 more union disjoint since else could combine

For every interval containing a point p in the open set S, take the union of all such intervals that are also contained in U. These intervals cover all $p \in S$ and are disjoint since else an overlapping interval can be combined into 1 larger interval, providing a maximal disjoint union U that covers the open set. Now we show its countable since each interval can be mapped to a index j and since each disjoint set is maximal and $\mathbb{Q} \cap U$ is countable due to being an ~~only~~ enumeration of rational #s. then the index array I must be countable, so there exists a sequence disjoint intervals to construct any open set.

4. Prove: If $S_1 = \bar{S}$ and $S_2 = \bar{S}_1$, then $S_1 = S_2$

Given $\bar{S} = \{p \in X$ s.t. $\exists$ subseq p_n in S converging to $p\}$

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~~Since S_1 is the closure of S, S_1 must be a closed set. Therefore $\bar{S}_1 = S_1$ since the closure of a closed set is itself. Thus $S_1 = S_2$.~~

~~If $\bar{S}$ is the closure of S, then $\bar{S}$ must contain all ~~subsequential~~ limits of convergent sequences in S. Now lets take $\bar{S}$, and we WTS that there cannot exist a p of S subseq limit of seq in that is not contained by $\bar{S}$. Suppose for contradiction that $\exists$ such a $p_n \to p$. p_n must contain at least 1 subsequential limit from S or else p_n was already~~

$\exists \epsilon > 0$ s.t. $\forall k > p - \epsilon$, $k \notin \bar{S}$ [ ] [ ] [......] $+ d(p, p_n) < \epsilon$

S_n $\qquad \forall \epsilon > 0 \qquad$ $\sup(S)$ in S $\qquad$ S_n $d(p,m) > \epsilon$ $> d(p, p_n)$

p_n

~~contained in S, so p ∈~~ $\bar{S}$. However, since p is a ~~subsequential~~ limit of $\bar{S}$, there $\exists N$ s.t. $\forall k > N$, $|p_{n_k} - p| < \epsilon$. But since p is not a subsequential limit of S, we know there doesn't exist a subseq of S that converges to p, so $p - \epsilon > m$ for all $m \in S$. Thus all $m < p - \epsilon < p_{n_k}$, so $\forall S_{(n_k)}$ which would be subsequential limits of $\bar{S}$ are $>$ max of S, and since we know $\sup(S) \geq$ all subseq limits of S, we know $\exists \epsilon > 0$ s.t. $p_{n_k} + \epsilon \leq \sup(S)$ for at least one n_k, so $S_{n_k} \leq \sup(S) - \epsilon <$ max of S, a contradiction

5. Prove $\bar{S} = \bigcap \{U : U \in X, S \subseteq U\}$ (closed subsets in X)

We'll call $\{U\}$ all the closed subsets in X containing S.
WTS $\bar{S} \subset \bigcap \{U\}$. We know $S \subset \bigcap\{U\}$ since each subset must contain S already, so we just need to show that each U in $\{U\}$ also contains all the limits p of convergent sequences p_n in S. ~~Since each U is closed, we know S\U is open so that ∃r>0 s.t. there's an open ball B_r(p) for every p ∈ S\U, B_r(p) ∈ S\U.~~
Since we know $\bigcap\{U\}$ is closed, $\overline{\bigcap\{U\}} = \bigcap\{U\}$, and since $S \subset \bigcap\{U\}$ the closure of $\bigcap\{U\}$ must contain all limits p of conv seq in S, by def of closure, which is still just $\bigcap\{U\}$, so we now have $\bar{S} \subset \bigcap\{U\}$

4. WTS there cannot exist lim of seq in $\bar{S}$ not contained in $\bar{S}$ (that isn't in $\bar{S}$).
Suppose for contradiction $\exists$ a sequence p_n in $\bar{S}$ that $p_n \to p$, meaning $\forall \epsilon > 0$ $\exists N$ s.t. $d(p_n, p) < \frac{\epsilon}{2}$. Because p isn't in $\bar{S}$, it's not a limit of a seq in S, so $d(p, m) > \epsilon$ for all $m \in S$. We also know $\exists k \in S$ s.t. $d(p_n, k) > \frac{\epsilon}{2}$ since p_n contains a limit of a seq in S or elements of S itself, so $d(p,k) < d(p_n, p) + d(p_n, k) < \epsilon < d(p, m)$, which is impossible since for some $m \in S$, $m = k$, which invalidates the inequality, the contradiction.