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Math 104 Hw 6

1. Since $[0,1]$ is sequentially compact then for every sequence a_n in $[0,1]$, $\exists a_{n_k}$ that converges to $a \in [0,1]$ as $k \to \infty$. We can then create another sequence $b_n \in [0,1]$, $\exists$ $b_{(n_k)}$ has it's own subsequence $b_{n_{k_m}}$ that converges to b. Since $a_{n_k} \to a$, then $a_{n_{k_m}} \to a$, so thus $(a_{n_{k_m}}, b_{n_{k_m}}) \to (a,b)$, so $[0,1]^2$ is compact.

2. If decimal expansion is only 4 or 7 this is the same as ~~calc~~ seeing if the set of infinite length bitstrings are countable. This isn't countable since if we have a hypothetical enumeration of bitstrings $a_1 = .1011\ldots$
$a_2 = .0101$

we can take the diagonal and flip the bits, creating an infinite length bitstring differing from each a_i by at least 1 elem.

E is compact since it's a closed subset of $\mathbb{R}$ which is compact. E is a subset of $\mathbb{R}$ obviously since $0 \le E < 1$. E is closed since suppose for sake of contradiction there is a limit e of a seq. e_n in E, that is not in E, meaning there is at least 1 digit that's not a 4 or 7. Lets take the first different digit that occurs called d. WLOG suppose $e >$ the nearest E member e', so to make e as close as possible to e', we have $e = (\text{first } k \text{ digits of } e' + d \times 10^{-k})$. Thus $e - e' = .A\ldots d00$ (to make as close as possible), we
$.A\ldots d'77\ldots$

can create an $\epsilon < 10^{-k+1}$ so $e - e' > \epsilon$, so there can be no limit e existing, so E is closed, so E is compact.

$$\overline{\bigcup_i A_i} = \bigcup_i \overline{A_i}$$

3. $A_1, A_2 \in A$ $\quad B = \bigcup_i A_i$

Suppose we have A_i represent a distinct rational number, for i mapping to all rat'ls. Thus $\overline{A_i} = A_i$. Then $B = \bigcup A_i = \mathbb{Q}$ and $\overline{B} = \overline{\mathbb{Q}} = \mathbb{R}$ since for every irrational i we can find 2 r's infinitely close to i and continue finding infinite rationals between the r pairs, allowing use to have a sequence of r's converging to i, so $\overline{\mathbb{Q}} = \mathbb{R}$. Thus $\overline{B} = \mathbb{R} \supseteq \mathbb{Q} = \bigcup \overline{A_i}$

4. An example of a closed set which not a countable union of closed intervals is the set E from Q2. We proved it's a closed set since it's compact and it must be made of an uncountable union of closed intervals since the only valid closed intervals are single element intervals i.e. $[a,a]$ since else there will exist a number between the bounds not in E. Thus there is a closed interval needed for all members of E, which we proved is uncountable.

The argument is wrong since it doesn't prove that the union of closed intervals must be countable, which is disproven since there exists a closed subset of $\mathbb{R}$ (E) which is an uncountable union of closed intervals