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$\frac{3}{4n-2} = \varepsilon$

$\frac{3}{4\varepsilon} + \frac{1}{2}$

Math 104 Hw 8

1. $f_n(x) = \frac{n+\sin x}{2n+\cos n^2 x}$ We can bound $f_n(x)$ as

$\frac{n-1}{2n+1} \leq f_n(x) \leq \frac{n+1}{2n-1}$ and as n gets large this reduces to

We want to show that $f_n(x) \to f(x) = \frac{1}{2}$, so we can show that $\frac{1}{2} - \frac{n-1}{2n+1} = \frac{(n-1)-(n+\frac{1}{2})}{2n+1} = \frac{+\frac{3}{2}}{2n+1} = \frac{+3}{4n+2}$ and $\frac{n+1}{2n-1} - \frac{1}{2} = \frac{\frac{1}{2}}{2n-1} = \frac{3}{4n+2}$, so

$\frac{3}{4n+2} \leq |f_n(x) - \frac{1}{2}| \leq \frac{3}{4n-2}$, so for $\forall \varepsilon > 0$, we can find an $N > \frac{3}{4\varepsilon} + \frac{1}{2}$, so $|f_N(x) - \frac{1}{2}| \leq \varepsilon$, so $f_n(x)$ is uniformly convergent

2. $f(x) = \sum_{n=1}^{\infty} a_n x^n$ WTS: if $\sum_n |a_n| < \infty$, then series is continuous $[-1,1]$
We can use the Weierstrass M-test which says for a convergent series $\sum_{n=1}^{\infty} u_n(x)$ that there's a convergent series of constants $\sum_{n=1}^{\infty} M_n$ that $|u_n(x)| \leq M_n$, then the series is absolutely + uniformly convergent.

Since $\sum |a_n| < \infty$ means a_n is absolutely convergent, so if we let $M_n = |a_n|$, then we know x^n on $[-1,1]$ is $-1 \leq x^n \leq 1$, and thus $|u_n| = |a_n x^n| < |a_n| = M_n$ since $-|a_n| < |a_n x^n| < a_n$, so the series $\sum_{n=1}^{\infty} a_n x^n$ is uniformly convergent, so $a_n x^n$ is continuous. Then, $a_n = n^{-2}$ satisfies $\sum_n |a_n| < \infty$ since $\frac{1}{n^2}$ is absolutely convergent since $\frac{1}{n^2} = |\frac{1}{n^2}|$ which is convergent from the p-test, so we can apply the first part now to show $\sum_{n=1}^{\infty} n^{-2} x^n$ is continuous.

3. $f(x) = \sum_n x^n$ To show $f(x)$ on $(-1,1)$ is continuous, we WTS uniform convergence on $[-a,a]$ for $0 < a < 1$. We show this via Weierstrass M-test where for $\sum_n a^n = a^1 + a^2 \ldots a^n$ which is an infinite geometric series as $n \to \infty$, so $\frac{a}{1-a}$ which is a constant/convergent, so if we have $M_n = a^n$, we show $\sum_n a^n$ is uniformly + absolutely convergent on $a \in (0,1)$, which extends to $(-1,1)$ from absolute convergence. To show $\sum_n x^n$ isn't uniformly convergent we WTS

that $\exists\, \epsilon > 0$ s.t. $\forall\, N > 0$, $\exists\, x \in (-1, 1)$ and $\exists\, n > N$ s.t. $|f_n(x) - f(x)| > \varepsilon$: $f_n(x) = \Sigma_{i=0}^{n} x^i = \frac{1-x^n}{1-x}$ since $x \in (-1,1)$

, so $|f_n(x) - f(x)| = \left|\frac{1-x^n}{1-x} - \frac{1}{1-x}\right| = \left|\frac{-x^n}{1-x}\right|$ Wts $|-x^n/(1-x)| > \varepsilon$, where ε is a constant. Suppose $x \geq a, \in (0,1)$, so $|x^n/(1-x)| \geq |a^n/(1-x)| > \varepsilon$. if $x < 1 - \frac{1}{\varepsilon}a^n$ then, $|a^n/(1-x)| > |a^n/\varepsilon a^n| = \varepsilon$, so $|x^n/(1-x)| \geq |a^n/(1-x)| > \varepsilon$, so $f(x)$ doesn't have uniform convergence.