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Taylor's: $P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$

$\exists\, x \in (\alpha, \beta)$ s.t.

$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n$

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Rudin

5.4. $C_0 + C_1 x + \dots C_{n-1} x^{n-1} + C_n x^n = f'(x)$ when
$f(x) = C_0 \cdot x + \frac{C_1}{2} x^2 \dots + \frac{C_n}{n+1} x^{n+1}$

$f(1) = C_0 + \frac{C_1}{2} + \dots \frac{C_n}{n+1} = 0 = f(0)$. Thus, since f is continuous and differentiable from $(0,1)$ and $f(1) = f(0)$ Then $f'(x) = 0$ for some real root $x \in [0,1]$

5.8. $f' =$ cont and $\varepsilon > 0$. WTS $\exists\, \delta > 0$ s.t. if $|t-x| < \delta$, $a \le x \le b$, $a \le t \le b$

We know $\frac{f(t)-f(x)}{t-x} = f'(c)$ for some $a < c < b$ since from the Mean Value Theorem, since we know this is true since f' is cont, so f is cont and differentiable
Thus we have $|f'(c) - f'(x)| < \varepsilon$ from the continuousness of $f(c)$ assuming the existence of δ s.t. $|c-x| < \delta$ which we can take from $|t-x| < \delta$ letting $t = c$

5.18. $f^{(n-1)}$ exists for $t \in [a,b]$

Define: $Q(t) = \frac{f(t)-f(\beta)}{t-\beta} \rightarrow f(t) - f(\beta) = (t-\beta) Q(t)$

WTS: $f(\beta) = P(\beta) + \frac{Q^{n-1}(\alpha)}{(n-1)!} \cdot (\beta-\alpha)^n$

Induction: Base Case: $P(\beta) = \sum_{k=0}^{0} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k$, We know $Q^0(\alpha) = \frac{f(\alpha)-f(\beta)}{\alpha-\beta}$ for $n=1$

$f(\beta) = f(\alpha) + \frac{f(\alpha)-f(\beta)}{\alpha-\beta} \cdot (\beta-\alpha) \rightarrow 0 = 0$ ✓

Inductive Step: Can assume $f(\beta) = P(\beta) + \frac{Q^{n-1}(\alpha)}{(n)!} \cdot (\beta-\alpha)^n$

Now, for $n+1$ we have: $f(\beta) = \sum_{k=0}^{n} \frac{f^k(\alpha)}{k!}(\beta-\alpha)^k + \frac{Q^n(\alpha)}{(n+1)!}(\beta-\alpha)^{n+1}$

We know $f'(t) = (t-\beta) Q'(t) + Q(t) \rightarrow f^k(t) = k\, Q^{k-1}(t) + (t-\beta) Q^k(t) \rightarrow$

$f(\beta) =$

$$f(B) = \frac{f^n(\alpha) - n\,Q^{n-1}(\alpha)}{(\alpha - B)(n+1)!} \cdot (B-\alpha)^{n+1} + \sum_{k=0}^{n} \frac{(B-\alpha)^k}{k!} \cdot \left(k\,Q^{k-1}(\alpha) + (\alpha - B)\,Q^k(\alpha)\right)$$

$$= P(B) + \frac{Q(\alpha)}{0!}(B-\alpha) - \frac{Q^n(\alpha)}{n!}(B-\alpha)^{n+1}$$

$$= \sum_{k=0}^{n} \frac{f^k(\alpha)}{k!}(B-\alpha)^k + \frac{Q^n(\alpha)}{(n+1)!}(B-\alpha)^{n+1}$$ which is true by ind. hypothesis

#1) $f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \le 0 \end{cases}$ This is infinitely differentiable ~~since~~ proven from example 3.

We then want to make use of $f(1-x)$ since then we can use $g(x) = \frac{f(x)}{f(x)+f(1-x)}$, since $g(x) = 0$ if $x \le 0$ since $f(1-x)$ will be nonzero. and if $x \ge 1$, then $g(x) = 1$ since $f(1-x) = 0$. We finally know that $g(x)$ is infinitely differentiable (smooth) since $f(x)$ is infinitely differentiable and $f(1-x)$ also is infinitely differentiable, since you can just reflect $f(x)$ and shift it to obtain $f'(x)$, not changing the shape. Also $f(x)+f(1-x) \neq 0$ since one must always be nonzero.

5.22) a) Suppose that f has more than 1 pt (2 pts, x, y) that are fixed. Thus $f(x) = x$, $f(y) = y$. Since f is differentiable everywhere, it's also continuous so by MVT, $\exists\, c \in (x,y)$ s.t. $f'(c) = \frac{f(x)-f(y)}{x-y} = 1$, which is a contradiction to the $f'(t) \neq 1$.

b) Suppose $f(t) = t + (1-e^t)^{-1} = t \Rightarrow (1-e^t)^{-1} = 0$, meaning $1-e^t$ must approach $\pm\infty$ for $f(t) = t$, which cannot happen for all $t \in \mathbb{R}$.

c) Suppose $\exists\, A < 1$ s.t. $|f'(t)| \le A$. We know that $|x_3 - x_2| \le |A||x_2 - x_1|$ from $|f(x_2) - f(x_1)| \le |A||x_2 - x_1|$ since $|f'(t)| \le A$ for all t. Thus we can generalize to $|x_{n+1} - x_n| \le |A|^{n-1}|x_2 - x_1|$. Thus x_n must converge since $|A| < 1$. Thus, as $n \to \infty$, $x_n = x_{n+1} = f(x_n) = x = f(x)$, so x is a fixed point.

d) If we define the points as in the zig-zag, the even points are our goal points and our odd point are our actual $(x_n, f(x_n))$ relationship. Thus as $n \to \infty$, the actual points get closer and closer to approximated points as $x_n = x_{n+1} = f(x_n)$. This is visually shown since the rise will always be lower than run, so eventually the gap will be closed.