Math 104 Hw 1

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1.10. Prove $(2n+1)+(2n+3)+\ldots(4n-1)=3n^2$ for all positive integers n.

Using sum of arithmetic sequence formula (avg term · # of terms = sum), we have

$$\left(\frac{(2n+1)+(4n-1)}{2}\right)\cdot\left(\frac{\text{range of seq.}}{\text{step size}}+1\right) = 3n\cdot\left(\frac{(4n-1)-(2n+1)}{2}+1\right)$$
$$= 3n\cdot(n-1+1) = \boxed{3n^2}$$

1.12. a) $n=1$ $\quad (a+b)^1$ vs. $\binom{1}{0}a^1+\binom{1}{1}b^1$

$a+b$ vs. $\frac{1!}{1!0!}a^1+\left(\frac{1!}{1!0!}\right)b^1$

$a+b=a+b$ ✓

$n=2$ $\quad (a+b)^2$ vs. $\binom{2}{0}a^2+\binom{2}{1}ab+\binom{2}{2}b^2$

$a^2+2ab+b^2 = a^2+2ab+b^2$ ✓

$n=3$ $\quad (a+b)^3$ vs. $\binom{3}{0}a^3+\binom{3}{1}a^2b+\binom{3}{2}ab^2+\binom{3}{3}b^3$

$a^3+3a^2b+3ab^2+b^3$ vs. $a^3+3a^2b+3ab^2+b^3$ ✓

b) $\binom{n}{k}+\binom{n}{k-1}$

$$= \frac{n!}{k!(n-k)!}+\frac{n!}{(k-1)!(n-k+1)!} = n!\left(\frac{n-k+1}{k!(n-k+1)!}+\frac{k}{k!(n-k+1)!}\right) = n!\left(\frac{n+1}{k!(n-k+1)!}\right)$$
$$= \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k} \checkmark$$

c) Base case: $n=1$ $\quad (a+b)^1 = \binom{1}{0}a^1+\binom{1}{1}b^1 = a+b$

Inductive Hypothesis: For $n\in\mathbb{N}$, $(a+b)^n = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i$

Inductive Step: $(a+b)^{n+1} = (a+b)(a+b)^n = (a+b)\sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i$

$$= \sum_{i=0}^{n}\binom{n}{i}a^{n-i+1}b^i + \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^{i+1}$$

$$= a^{n+1} + \sum_{i=1}^{n}\binom{n}{i}a^{n-i+1}b^i + b^{n+1} + \sum_{i=0}^{n-1}\binom{n}{i}a^{n-i}b^{i+1} = a^{n+1}+b^{n+1}+\sum_{i=1}^{n}\left(\binom{n}{i}+\binom{n}{i-1}\right)a^{n-i+1}b^i$$

$= a^{n+1} + b^{n+1} + \sum_{i=1}^{n} \binom{n+1}{i} a^{n-i+1} b^i$

$= \sum_{i=0}^{n+1} \binom{n+1}{i} a^{(n+1)-i} b^i \Rightarrow$ conclusion of inductive step

2.1. $\sqrt{3} \rightarrow$ root of polynomial $x^2-3=0$. By Rational Root Thm, the only rational potential solutions would be $\pm 1, \pm 3$, none of which satisfy the equation, so $\sqrt{3}$ is not rational

Similarly, $\sqrt{5}$ is root of polynomial $x^2-5=0$ with potential rational solutions $\pm 1, \pm 5$
$\sqrt{7}$ " $x^2-7=0$ " $\pm 1, \pm 7$
$\sqrt{24}$ " $x^2-24=0$ " $\pm 1, 2, 3, 4, 6, 8, 12, 24$
$\sqrt{31}$ " $x^2-31=0$ " $\pm 1, 31$

None of these are valid solutions to their respective equations so $\sqrt{5}, \sqrt{7}, \sqrt{24}, \sqrt{31}$ all are not rational

2.2. Similarly, $\sqrt[3]{2}$ is root of polynomial $x^3-2=0$ w/ potential rational solutions $\pm 1, 2$
$\sqrt[7]{5}$ " $x^7-5=0$ " $\pm 1, 5$
$\sqrt[4]{13}$ " $x^4-13=0$ " $\pm 1, 13$

None are valid solutions to respective equations, so $\sqrt[3]{2}, \sqrt[7]{5}, \sqrt[4]{13}$ aren't rational

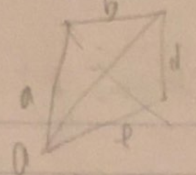
2.7. a) $\sqrt{4+2\sqrt{3}} - \sqrt{3}$ $\quad n = \sqrt{4+2\sqrt{3}}$ $\quad n^2 = 4+2\sqrt{3} = 1+2\sqrt{3}+3 = (1+\sqrt{3})^2 \Rightarrow n = 1+\sqrt{3}$

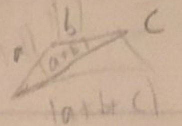
$\sqrt{4+2\sqrt{3}} - \sqrt{3} = n - \sqrt{3} = 1+\sqrt{3}-\sqrt{3} = \boxed{1}$

b) $\sqrt{6+4\sqrt{2}} - \sqrt{2}$ $\quad n = \sqrt{6+4\sqrt{2}}$ $\quad n^2 = 6+4\sqrt{2} = 2+4\sqrt{2}+4 = (\sqrt{2}+2)^2 \Rightarrow n = \sqrt{2}+2$

$\sqrt{6+4\sqrt{2}} - \sqrt{2} = n - \sqrt{2} = 2+\sqrt{2}-\sqrt{2} = 2$

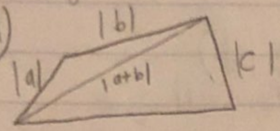
$|a+b| < |a| + |b|$





$|a+b| - |c| \geq$

3.6. a)



Suppose $d = a+b$, From triangle inequality we have $|a| + |b| \geq |a+b| = |d|$. Applying it to $e = a+b+c = d+c$ we have $|d| + |c| \geq |e|$

which reduces to $|a+b| + |c| \geq |a+b+c|$

and adding $|a| + |b| \geq |a+b|$

we get $|a| + |b| + |c| \geq |a+b+c|$

b) Base case: $n=2$ $|a_1 + a_2| \leq |a_1| + |a_2|$ via triangle inequality.

Inductive Hypothesis: $|a_1 + a_2 \ldots + a_n| \leq |a_1| \ldots + |a_n|$ for $a_1 \ldots a_n$

Inductive Step: For $n+1$ terms we can say $d = a_1 + \ldots a_n$, so by triangle inequality $|d + a_{n+1}| \leq |d| + |a_{n+1}|$

which simplifies to $|a_1 + \ldots + a_{n+1}| \leq |a_1 + a_2 \ldots + a_n| + |a_{n+1}|$

to which we add $|a_1 + \ldots a_n| \leq |a_1| + \ldots |a_n|$

to yield $|a_1 + \ldots a_{n+1}| \leq |a_1| + \ldots + |a_{n+1}|$, the conclusion of the inductive step

4.11 Given: $a, b \in \mathbb{R}$ where $a < b$. From the Denseness of $\mathbb{Q}$, we know there exists $a_1 \in \mathbb{Q} \subset \mathbb{R}$ that $a < a_1 < b$. Now we can suppose for contradiction that there are finite distinct rationals between a, b, lets say of size $k \rightarrow \{a_1, a_2, \ldots a_k\}$, where WLOG, $a_1 < a_2 < \ldots < a_k$. Now by Denseness of $\mathbb{Q}$, we ~~gener~~ know there exists a_{k+1} that $a_1 < a_{k+1} < a_2$, so we have $k+1$ ~~#~~ elements at least in the set. That's a contradiction so there's an infinite # of rationals between a, b.

$a+b$

$a \leq \sup(A+B)-b$

$a+b \leq \sup(A)+\sup(B)$

4.14 a) $A \subset \mathbb{R}$ $B \subset \mathbb{R}$ $b \leq \sup(B)$ $a \leq \sup(A)$

$a+b \leq \sup(A+B)$ by definition of sup for all a, b

$a \leq \sup(A+B)-b$, so all values of a are upper bounded by $\sup(A+B)-b$,

so $\sup(A) \leq \sup(A+B)-b \rightarrow b \leq \sup(A+B)-\sup(A)$, so

$\sup(B) \leq \sup(A+B)-\sup(A) \rightarrow \sup(A)+\sup(B) \leq \sup(A+B)$.

Summing $b \leq \sup(B)$ & $a \leq \sup(A) \Rightarrow a+b \leq \sup(A)+\sup(B)$, meaning $\sup(A+B) \leq \sup(A)+\sup(B)$ since a,b represent all values within A, B. Thus, from antisymmetry, we have $\sup(A+B) = \sup(A)+\sup(B)$

b) $\sup(S) = -\inf(-S)$ by and suppose $S = A+B$, we have $\sup(A+B) = -\inf(-A-B)$.

Substituting into part A, we have $-\inf(-A-B) = -\inf(-A) - \inf(-B)$, declaring $A' = -A$ and $B' = -B$, we have $\inf(A'+B') = \inf(A') + \inf(B')$, where A' and B' are still nonempty and are subsets of $\mathbb{R}$, so we can generalize this to all $A, B \in \mathbb{R}$.

7.5 a) $\lim s_n$ $s_n = \sqrt{n^2+1} - n \cdot \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n} = \boxed{0}$

b) $\lim(\sqrt{n^2+n}-n)$ $(\sqrt{n^2+n}-n)\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{\sqrt{1+0}+1} = \boxed{\frac{1}{2}}$

c) $\lim(\sqrt{4n^2+n}-2n)$ $(\sqrt{4n^2+n}-2n)\frac{\sqrt{4n^2+n}+2n}{\sqrt{4n^2+n}+2n} = \frac{n}{\sqrt{4n^2+n}+2n} = \frac{1}{\sqrt{4+\frac{1}{n}}+2} = \boxed{\frac{1}{4}}$