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Math 104 Hw 2

9.9. a) Given that $\lim s_n = +\infty$, we know for all $C \in \mathbb{R}$, there exists an $N > N_0$ that for all $n > N$, $s_n > C$. Since, $s_n \leq t_n$ for all $n > N > N_0$, we know for all n that $t_n \geq s_n > C$, so $t_n > C$, meaning $\lim t_n = +\infty$.

b) Given that $\lim t_n = -\infty$, we know for all $C \in \mathbb{R}$ there exists an $N > N_0$ that for all $n > N$, $t_n < C$. Since $s_n \leq t_n$ for all $n > N > N_0$, we know that $C > t_n \geq s_n$, so $s_n < C$, meaning $\lim s_n = -\infty$

c) If $\lim s_n$ and $\lim t_n$ exist, then we know $\lim s_n = s$ and $\lim t_n = t$. We know $s_n \leq t_n$ for all $n > N_0$, so $s_n - t_n \leq 0$. We can construct $N_1 \geq N_0$ that for $n_1 \geq N_1$ $(s_{n_1} - t_{n_1}) - \epsilon < s - t$. Suppose for contradiction $s > t$, if any we $\epsilon > 0$ then construct $\epsilon = s_{n_1} - t_{n_1}$, we have $s < t$, a contradiction, so $\boxed{s \leq t}$.

9.15. $\lim_{n\to\infty} \frac{a^n}{n!} \leq \lim_{n\to\infty} \frac{a^n}{\frac{n}{2}^{\frac{n}{2}}}$ since $\frac{n}{2}^{\frac{n}{2}} \leq \prod_{i=n/2}^{n} i \leq n!$. $\;= \left(\frac{n!}{(n/2)!}\right)$

$\frac{a^n}{\frac{n}{2}^{\frac{n}{2}}} = \frac{a^n}{\sqrt{\frac{n}{2}}^n} = \left(\frac{a}{\sqrt{\frac{n}{2}}}\right)^n$ $\quad \lim_{n\to\infty} \left(\frac{a}{\sqrt{n/2}}\right)^n = 0^n = 0$, so therefore $\lim_{n\to\infty} \frac{a^n}{n!} \leq 0$, and since $a, n \geq 0$, $\lim_{n\to\infty} \frac{a^n}{n!}$ must be 0.

10.7. Given: $\sup(S) > \max(S)$ We know that $\sup(S) - \epsilon < s$ for some s in S. Expressing s as $\frac{1}{n}$, where $n \in \mathbb{N}, > 0$, we get $\sup(S) - \frac{1}{n} < s$. For all $\epsilon > 0$ there's an $N > \frac{1}{\epsilon}$, so we know any $n \geq N$ allows $n > \frac{1}{\epsilon}$, so we now know $\epsilon > \frac{1}{n} > \sup(S) - s_n = |$
$\Rightarrow |\sup(S) - s_n| < \frac{1}{n} < \epsilon \Rightarrow \lim s_n = \sup(S)$

$\frac{1}{4}\cdot\left(\frac{1}{3}\cdot\left(\frac{1}{2}\cdot 1^2\right)^2\right)^2$ $\prod_{i=1}^{N}\left(\frac{1}{N-i}\right)^{2^i}\sigma_n$ $-2<1$

10.8. $\sigma_n = \frac{1}{n}(s_1+s_2 \dots + s_n)$ when all $s_i > 0$ and $s_{i+1} \geq s_i$

Base Case: $n=1$ $\sigma_1 = 1 \cdot s_1 = s_1$ s_1 is the only term, so increasing

Hypothesis: (σ_n) is an increasing sequence, meaning $\sigma_n \geq \sigma_{n-1} \dots \geq \sigma_1$

Inductive Step: $\sigma_{n+1} = \frac{1}{n+1}(s_1+s_2 \dots + s_{n+1})$ $\sigma_n = \frac{1}{n}(s_1 + \dots s_n)$

$$= \frac{n \cdot \sigma_n + s_{n+1}}{n+1}$$

Suppose $\sigma_{n+1} < \sigma_{n+1}$ for contradiction

$$\frac{n \cdot \sigma_n + s_{n+1}}{n+1} < \sigma_n$$

$$s_{n+1} < \sigma_n$$

$$s_{n+1} < (s_1 + \dots s_n)/n < (\overbrace{s_1 + \dots s_1}^{n})/n$$

$s_{n+1} < s_n$, which is impossible, so by contradiction

$\sigma_{n+1} \geq \sigma_n$, so (σ_n) is an increasing sequence

10.9. $s_1 = 1$, $s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2$ for $n \geq 1$

a) $s_2 = \frac{1}{2}(1) = \frac{1}{2}$ $s_3 = \frac{2}{3}\left(\frac{1}{2}\right)^2 = \frac{1}{6}$ $s_4 = \frac{3}{4} \cdot \left(\frac{1}{6}\right)^2 = \frac{1}{48}$

b) Base case: $n=2$ $s_2 = \frac{1}{2}$ $s_2 > 0$ and $s_n \leq 1$

Hypothesis: $s_n > 0$ and $s_{n+1} \geq s_1$

Inductive Step: $s_{n+1} = \frac{n}{n+1}(s_n^2)$, since $\frac{n}{n+1} > 0$ and $s_n > 0$, $s_{n+1} > 0$.

Also, since $0 < s_n \leq 1$, $\frac{n}{n+1}(s_n^2) \leq s_n^2 \leq s_n \leq 1$, so

s_{n+1} is decreasing and lower-bounded, so it exists

c) $\lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \frac{n}{n+1} s_n^2 = \lim_{n\to\infty} \frac{n}{n+1} \cdot \lim_{n\to\infty} s_n^2 = s \rightarrow 1 \cdot s^2 = s$ $s = 1, 0$,

since s_1 is decreasing from 1, s must be 0.

10.10. $s_1 = 1$, $s_{n+1} = \frac{1}{3}(s_n+1)$ for $n \geq 1$

a) $s_2 = \frac{1}{3}(1+1) = \frac{2}{3}$ $s_3 = \frac{1}{3}(\frac{2}{3}+1) = \frac{5}{9}$ $s_4 = \frac{1}{3}(\frac{5}{9}+1) = \frac{14}{27}$

b) Base Case: $s_1 = 1$, $s_1 > \frac{1}{2}$

Inductive Hypothesis: $s_n > \frac{1}{2}$

Inductive Step: $s_{n+1} = \frac{1}{3}(s_n+1)$ $3s_{n+1} - 1 = s_n > \frac{1}{2}$ $s_n > \frac{3}{2}/3 \Rightarrow \boxed{s_n > \frac{1}{2}}$

c) WTS: (s_n) is decreasing, that $s_{n+1} \leq s_n$

Base Case: $s_2 \leq s_1 \rightarrow \frac{2}{3} \leq 1$ ✓

Inductive Hypothesis: $s_n \leq s_{n-1}$

- Inductive Step: $s_{n+1} = \frac{1}{3}(s_n+1) \rightarrow \frac{1}{3}(s_n+1) \leq s_n$ $s_n + 1 \leq 3s_n$

$1 \leq 2s_n \rightarrow s_n \geq \frac{1}{2}$, which is proved in b

d) s_n is decreasing and bounded above and below by $1, \frac{1}{2}$, so the limit exists.

$2s-3$

$s = \lim_{n\to\infty} s_n = 3(\lim_{n\to\infty} s_{n+1} - 1) = 3s - 1$

$2s = 1$ $\boxed{s = \frac{1}{2}}$

1

ε

$\frac{1}{4n^2} > 0$

10.11. a) $t_1 = 1 \quad t_{n+1} = \left[1 - \frac{1}{4n^2}\right] \cdot t_n$ for $n \geq 1$

Base case: $t_2 = \left(1 - \frac{1}{4}\right) \cdot 1 = \frac{3}{4} \quad t_2 > 0 \quad t_2 < t_1$

Hypothesis: $t_n > 0 \quad t_n < t_{n-1}$

Inductive Step: $t_{n+1} = \left[1 - \frac{1}{4n^2}\right] \cdot t_n$ We know $0 < \frac{1}{4n^2} < 1$ for $n > 1$, so

$0 < 1 - \frac{1}{4n^2} < 1$, so $t_{n+1} < t_n$ and $t_{n+1} > 0$, so

t_n is bounded below and decreasing

b) $\frac{1}{e}$: maybe.

12. Squeeze Test: Len a_n, b_n, c_n be 3 seq that $a_n \leq b_n \leq c_n$, $L = \lim a_n = \lim c_n$

$\lim a_n = L \Leftrightarrow$ means for all $\epsilon > 0$, there exists N than for $n > N$,

$|a_n - L| < \epsilon$, similarly $|c_n - L| < \epsilon$. We then know that

$-\epsilon < a_n - L < \epsilon$, $-\epsilon < c_n - L < \epsilon$

$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon \Rightarrow L - \epsilon < b_n < L + \epsilon \rightarrow |b_n - L| < \epsilon$,

so $\boxed{\lim b_n = L}$

$2 < 5$

$3 < 5$

$2 - 3 < 0$